A Generalization of Little's Theorem on Pfaffian Orientations

M. H. de Carvalho, C. L. Lucchesi, U. S. R. Murty

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Faculdade de Computação
Universidade Federal de Mato Grosso do Sul
Campo Grande, MS, Brasil

A Generalization of Little's Theorem on Pfaffian Orientations*

Marcelo H. de Carvalho† Cláudio L. Lucchesi† U. S. R. Murty‡

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Abstract

Little [11] showed that, in a certain sense, the only minimal non-Pfaffian bipartite matching covered graph is the brace $K_{3,3}$. Using a stronger notion of minimality than the one used by Little, we show that every minimal non-Pfaffian brick G contains two disjoint odd cycles C_1 and C_2 such that the subgraph $G - V(C_1 \cup C_2)$ has a perfect matching. This implies that the only minimal non-Pfaffian solid matching covered graph is the brace $K_{3,3}$. (A matching covered graph G is *solid* if, for any two disjoint odd cycles C_1 and C_2 of G, the subgraph $G - V(C_1 \cup C_2)$ has no perfect matching. Solid matching covered graphs constitute a natural generalization of the class of bipartite graphs, see [5].)

Keywords Perfect matchings, matching covered graphs, solid matching covered graphs, Pfaffian orientations.

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*Supported by CNPq, Brasil.
†FACOM-UFMS, Campo Grande, Brasil
‡University of Waterloo, Waterloo, Canada

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1 Introduction

An edge of a graph is *admissible* if there is a perfect matching of the graph that contains it. A graph is *matching covered* if it is a nontrivial connected graph in which each edge is admissible. Unless otherwise specified, all graphs considered in this paper are matching covered. For general graph-theoretical notation and terminology, we follow Bondy and Murty [1]; and the terminology we use that is specific to matching covered graphs is essentially the same as in the pioneering paper of Lovász [13], in the book *Matching Theory* by Lovász and Plummer [14], and in our papers [2], [3] and [4]. However, in some cases, we have chosen to adopt new notation and terminology; these will be introduced in due course.

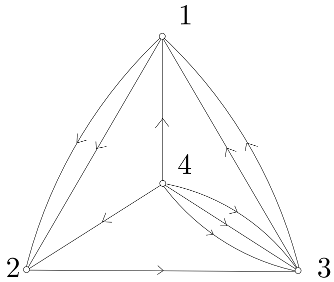
1.1 The Pfaffian orientation problem

Let D be an orientation of a matching covered graph G. With each perfect matching $M = \{e_1, e_2, \ldots, e_k\}$ of D, where, for $1 \le i \le k$, u_i and v_i denote, respectively, the tail and the head of e_i, we associate the permutation $\pi(M)$, where:

$$\pi(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & . & . & . & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & . & . & . & u_k & v_k \end{pmatrix}$$

The *sign* of M, denote by $\mathrm{sgn}(M)$, is the sign of the permutation $\pi(M)$. It can be seen that $\mathrm{sgn}(M)$ is independent of the order in which the arcs of M are enumerated. In the digraph shown in Figure 1, the permutations corresponding to all perfect matchings are odd, and hence all of them have negative sign. (Note that a necessary condition for all perfect matchings to have the same sign is that, for any two adjacent edges u and v, all edges are directed either from u to v, or all of them are directed from v to u.)

When all the perfect matchings of a digraph D have the same sign, it is known that the determinant of the adjacency matrix $\mathbf{A}$ of D is equal to the square of the number of perfect matchings of D. (The square root of the determinant of an even order skew-symmetric matrix $\mathbf{A}$ is known as the *Pfaffian* of $\mathbf{A}$. In the special case



$$\begin{bmatrix} 0 & 2 & -2 & -1 \\ -2 & 0 & 1 & -1 \\ 2 & -1 & 0 & -3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Figure 1: A digraph and its adjacency matrix

under consideration, the absolute value of the Pfaffian of $\mathbf{A}$ is the number of perfect matchings of D. See Lovász and Plummer [14, Chapter 8] for details.) For example, the digraph shown in Figure 1 has nine perfect matchings, and the determinant of its adjacency matrix is eighty-one.

Motivated by the above observation, a digraph is called *Pfaffian* if all its perfect matchings have the same sign. In the same vein, an (undirected) graph is *Pfaffian* if it admits an orientation that is Pfaffian. (It should be noted that, although the signs of perfect matchings of a digraph do depend on the chosen enumeration of its vertices, the property of the digraph being Pfaffian or non-Pfaffian is independent of that enumeration. More generally, two isomorphic digraphs are either both Pfaffian or both non-Pfaffian.) The digraph shown in Figure 1 is Pfaffian, but the digraph obtained from it by reversing the orientation of the edge 23, and leaving the orientations of all other edges as they are, is not Pfaffian.

The above definitions lead to the following important decision problems:

Problem 1.1 (The Pfaffian Recognition Problem)
Given a digraph D, decide whether D is Pfaffian.

Problem 1.2 (The Pfaffian Orientation Problem)
Given a graph G, decide whether G has a Pfaffian orientation.

Surprisingly, it is known that Problems 1.1 and 1.2 are polynomially equivalent [6, 21]. There are three important special classes of graphs for which these problems are known to be in $\mathcal{P}$.

Kasteleyn [10] showed that every planar graph is Pfaffian and described a polynomial-time algorithm for finding a Pfaffian orientation of a planar graph.

Little [11] showed that the Pfaffian Recognition Problem is in co-$\mathcal{NP}$ (see Theorem 4.2) for bipartite graphs. Several years later, McCuaig [15] and, independently, Robertson, Seymour and Thomas [19], showed that, for bipartite graphs, this problem is in $\mathcal{P}$. (Their work is of major significance because it is related to a number of fundamental, and seemingly unrelated, problems in algorithmic graph theory.)

A graph G is *near-bipartite* if it is matching covered, not bipartite but it has a pair of edges whose removal yields a bipartite matching covered graph. Fischer and Little [9] showed that the Pfaffian Recognition Problem is in co-$\mathcal{NP}$ for near-bipartite graphs. Recently, Miranda and Lucchesi [17] discovered a polynomial algorithm to solve the Pfaffian problem for near-bipartite graphs.

In this paper we introduce the notion of a deletion-contraction minor of a matching covered graph, and establish the structural result concerning minor-minimal non-Pfaffian matching covered graphs mentioned in the abstract. As a consequence, we deduce that the Pfaffian Recognition Problem is in co-$\mathcal{NP}$ for solid matching covered graphs. (We do not yet know if it is in $\mathcal{NP}$.)

1.2 Conformal subgraphs

A subgraph H of a graph G is *conformal* if $G - V(H)$ has a perfect matching. (Conformal subgraphs are called nice subgraphs in [14], well-fitted subgraphs in [15] and central subgraphs in [19].) As an immediate consequence of this definition, it follows that if F is a conformal subgraph of H and H is a conformal subgraph of G, then F is a conformal subgraph of G. The notion of a conformal subgraph, as we shall now explain, may be used to provide two useful alternative definitions of a Pfaffian graph.

Let D be a digraph and let T be a trail of even length in D. Regardless of the sense of traversal of T, the number of forward arcs and the number of reverse arcs have the same parity. We say that T is *evenly oriented* if the number of forward arcs is even and *oddly oriented* otherwise. For example, in the digraph shown in Figure 2, the cycle $(1, 2, 3, 4, 1)$ is evenly oriented whereas the cycle $(1, 4, 5, 6, 1)$ is oddly oriented.

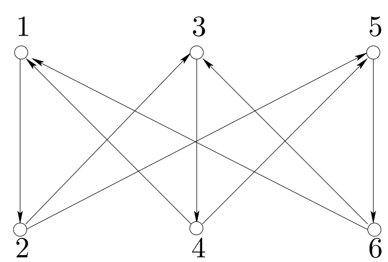


Figure 2: An orientation of $K_{3,3}$

The following basic result, relating the signs of two perfect matchings, is proved in Lovász and Plummer's book [14, Lemma 8.3.1].

LEMMA 1.3
Let M and N be two perfect matchings of a directed graph D and let ℓ denote the number of (M, N)-alternating cycles that are evenly oriented. Then, $\operatorname{sgn}(M)\operatorname{sgn}(N) = (-1)^{\ell}$.

Corollary 1.4
Let D be a directed graph and let M a perfect matching of D. Then, D is Pfaffian if and only if each M-alternating cycle of D is oddly oriented.

Corollary 1.5
A digraph D is Pfaffian if and only if each conformal cycle in D is oddly oriented.

In light of the above corollary, one may deduce that the digraph D in Figure 2 is not Pfaffian simply from the fact that the cycle $(1, 2, 3, 4, 1)$ is conformal and evenly oriented. This, of course, does not immediately imply that $K_{3,3}$ is non-Pfaffian. However, $K_{3,3}$ is non-Pfaffian and, indeed, it is the smallest non-Pfaffian matching covered graph. The following proposition may be verified easily.

Proposition 1.6
A matching covered graph G is Pfaffian if and only if each of its conformal subgraphs is Pfaffian.

2 Cuts, Contractions and Splicings

Let G be a connected graph. For any set X of vertices of G, we denote the coboundary of X by $\partial_G(X)$. Thus, $\partial_G(X)$ consists precisely of those edges that have one end in X, and one end in the complement $\overline{X}$ of X. If G is understood, we write simply $\partial(X)$ instead of $\partial_G(X)$. The set $\partial(X)$ is called a *cut*, the sets X and $\overline{X}$ its *shores*. A cut is *odd* if both its shores have an odd number of vertices and is *trivial* if one of its shores is a singleton.

Given a cut $C := \partial(X)$ of G, where X is a nonempty proper subset of V, the two graphs obtained by contracting X to a single vertex x and $\overline{X}$ to a single vertex $\overline{x}$ are denoted, respectively, by $G/X \to x$ and $G/\overline{X} \to \overline{x}$, and are called the C-*contractions* of G. If the names of the vertices resulting from contractions are not relevant, we simply denote the two C-contractions by G/X and $G/\overline{X}$, respectively. A graph G is the *splicing* of two graphs G_1 and G_2 if it has a cut C such that G_1 and G_2 are isomorphic to the two C-contractions of G and we refer to cut C as the *splicing cut* of G. The following assertion may be verified easily.

Proposition 2.1
Any splicing of two matching covered graphs is also matching covered.

The graph shown in Figure 3(a) is obtained by splicing two K_4's and the graph shown in Figure 3(b) by splicing a $K_{3,3}$ and a K_4. In each case, the associated splicing cut C is indicated by a thick line. (The graph in Figure 3(a) is the triangular prism and is usually denoted by $\overline{C_6}$.)

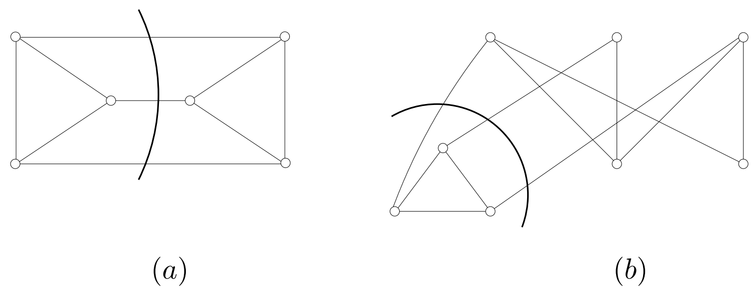


Figure 3: (a) $\overline{C_6}$: a splicing of two K_4's; (b) a splicing of a $K_{3,3}$ and a K_4

2.1 Separating cuts and tight cuts

Let G be a matching covered graph, C an odd cut of G. We say that C is *separating* if both C-contractions of G are matching covered. For, example, the two cuts shown in Figure 3 are separating cuts of the respective graphs. The following result characterizes separating cuts and is easy to prove.

Proposition 2.2
Let G be a matching covered graph. A cut C of G is separating if and only if every edge of G lies in a perfect matching that contains precisely one edge in C.

A cut is *tight* if every perfect matching of G has precisely one edge in the cut. Every tight cut is separating, but the converse is not true. For example the cut of $\overline{C_6}$ shown in Figure 3(a) is separating but is not tight. Every trivial cut is tight. If G is free of nontrivial tight cuts then it is a *brace* if it is bipartite, a *brick* otherwise.

If graph G has a nontrivial tight cut C, we may decompose it into its two C-contractions. If, in turn, one of these graphs has a nontrivial tight cut C', it may be decomposed into its two C'-contractions. By repeatedly applying this procedure, called the *tight cut decomposition procedure*, we obtain a family of bricks and braces. Lovász proved the following remarkable result [13].

Theorem 2.3
Any two applications of the tight cut decomposition procedure produce the same family of bricks and braces, up to multiple edges.

We denote by $b(G)$ the number of bricks obtained by a tight cut decomposition of G. Graph G is a *near-brick* if $b(G) = 1$. Thus, every brick is a near-brick. If G is bipartite, then for every tight cut C of G we have that both C-contractions of G are bipartite. Thus, if G is bipartite then $b(G) = 0$.

2.2 Solid matching covered graphs

A matching covered graph G is *solid* if each of its nontrivial separating cuts is tight. Every bipartite matching covered graph is solid. The brick $\overline{C_6}$ is not solid, whereas brick K_4 is solid.

A number of special properties that are enjoyed by bipartite graphs are shared by the more general class of solid matching covered graphs. For example, we showed in [5] that bipartite matching covered graphs and solid near-bricks share the property that their perfect matching polytopes may be defined without using the odd set inequalities. In the same paper, we presented a proof of the following useful theorem:

Theorem 2.4 (Reed and Wakabayashi)
A brick G has a nontrivial separating cut if and only if it has two disjoint odd cycles C_1 and C_2 such that $G - (V(C_1)) \cup V(C_2))$ has a perfect matching.

A brick is *odd-intercyclic* if any two of its odd cycles have at least one vertex in common. By the above theorem every odd-intercyclic brick is solid. Möbius ladders M_n, $n \equiv 0 \pmod 4$, are examples of such bricks. Figure 4 shows the Möbius ladder M_8 (with a Pfaffian orientation). Not every solid brick is odd-intercyclic. For example, the brick S_8 shown in Figure 6 is solid, but it is not odd-intercyclic.

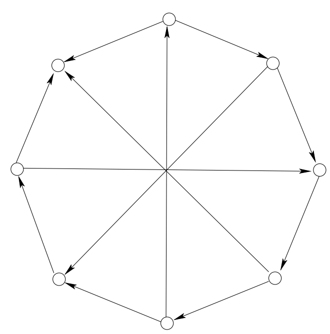


Figure 4: Möbius ladder M_8

No polynomial-time algorithm for recognizing solid bricks is known. It is not even known whether this problem is in $\mathcal{NP}$.

The main objective of this paper is to present a suitable generalization of Little's Theorem [11] concerning bipartite graphs to the class of all solid matching covered graphs. The following result is one of the essential ingredients of that generalization. Our proof is an adaptation of the proof given by Little and Rendl [12] of a special case of this result where the cuts under consideration are tight cuts rather than separating cuts.

Proposition 2.5
Let G be a matching covered graph. If G is Pfaffian then for any separating cut $C := \partial(X)$ of G, graph G has a Pfaffian orientation D such that the two C-contractions of D are also Pfaffian.

Proof: Let $G_1 := G/X \to x$ and $G_2 := G/\overline{X} \to \overline{x}$ denote the two C-contractions of G. By hypothesis, G has a Pfaffian orientation, say D_0. We shall describe a procedure for deriving a Pfaffian orientation D of G from D_0 and show that both C-contractions of D are Pfaffian, implying that G_1 and G_2 are also Pfaffian.

Let V_1 denote the set of vertices of $V(G_1) - x = \overline{X}$ that are incident with edges in C. Likewise, let V_2 denote the set of vertices of $V(G_2) - \overline{x} = X$ that are incident with edges of C. We now define a (possibly empty) subset W of $V_1 \cup V_2$ and show that the orientation D of G, obtained from D_0 by reversing the orientations of the edges in $\partial(W)$, is a Pfaffian orientation of G such that each D-contraction of G is also Pfaffian.

For this, let $e := v_1v_2$ denote an edge of C, where v_1 is its end in $\overline{X}$ and v_2 its end in X. Then, vertex v_1 lies in V_1 and vertex v_2 lies in V_2. By hypothesis, cut C is separating. Let M be a perfect matching of G such that $M \cap C = \{e\}$. For $i = 1, 2$, let M_i denote the restriction of M to G_i. Then, M_i is a perfect matching of G_i that contains edge e.

Let w_2 be any vertex of $V_2 - v_2$. See Figure 5. We now show that there exists in $G[X]$ an M-alternating path $P(w_2)$ that joins vertices v_2 and w_2. For this, note that as w_2 lies in V_2, then it is incident with an edge in C, say f. As C is separating, G has a perfect matching N such that $N \cap C = \{f\}$. Let Q be the M, N-alternating cycle in G that contains edge f. Then, Q meets C in precisely the two edges e and f. Let $P(w_2)$ denote the segment of Q in $G[X]$ that joins v_2 and w_2. Clearly, $P(w_2)$ is M-alternating. Likewise, for each vertex w_1 of $V_1 - v_1$, define path $P(w_1)$ to be an M-alternating path of $G[\overline{X}]$ that joins vertices v_1 and w_1. We now define W to be the subset of $(V_1 - v_1) \cup (V_2 - v_2)$ consisting of those vertices w such that $P(w)$ is oddly oriented in D_0. (Since D_0 is Pfaffian, note that the set W is independent of the choices of $P(w)$.)

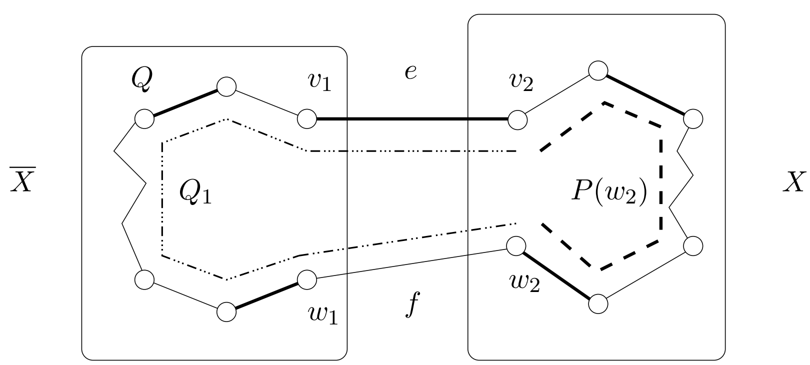


Figure 5: Proof of Proposition 2.5

Let D be the orientation of G obtained from the Pfaffian orientation D_0 of G by reversing the orientations on the edges of cut $\partial(W)$. Reversal of the orientations of the edges of a cut preserves the parity of every cycle of even length. As D_0 is Pfaffian, every M-alternating cycle of G is oddly oriented in D_0. Thus, every M-alternating cycle of G is oddly oriented in D. We deduce that D is a Pfaffian orientation of G. Moreover, as neither v_1 nor v_2 lies in W the reversal of the orientations of the edges of $\partial(W)$ preserves the parity of path $P(w)$ if and only if w does not lie in W. By definition, w lies in W if and only if $P(w)$ is oddly oriented in D_0. We deduce that $P(w)$ is evenly oriented in D, for each vertex w in $(V_1 - v_1) \cup (V_2 - v_2)$.

Let $D_1 := D/X \to x$ and $D_2 := D/\overline{X} \to \overline{x}$ be the two C-contractions of D. Then, for $i = 1, 2$, D_i is an orientation of G_i. To prove that D_1 is Pfaffian, let Q_1 denote any M_1-alternating cycle of G_1. We now show that Q_1 is oddly oriented in D_1. Firstly suppose that Q_1 is an M-alternating cycle of G itself. In that case, Q_1 is oddly oriented in D, whence it is also oddly oriented in D_1. We may thus assume that the edges of Q_1 do not constitute a cycle in G. See Figure 5. Then, Q_1 contains edge e and also an edge f of C whose end w_2 in X is distinct from v_2. Let W denote the cycle of G whose edge set is $E(Q_1) \cup E(P(w_2))$. Then, Q is M-alternating, whence oddly oriented in D. As $P(w_2)$ is evenly oriented in D, it follows that Q_1 is oddly oriented in D, whence oddly oriented in D_1. This conclusion holds for each M_1-alternating cycle Q_1 of G_1. We deduce that D_1 is a Pfaffian orientation of G_1. A similar reasoning may be used to prove that D_2 is a Pfaffian orientation of G_2. As asserted, D is a Pfaffian orientation of G whose C-contractions are also Pfaffian. $\square$

It should be noted that the converse of the above proposition is not valid. (For example, let G be the Petersen graph and let C be the cut consisting of a perfect matching of G. The two C-contractions of G are Pfaffian. But, G itself is not Pfaffian.) However, Little and Rendl [12] showed that the converse does hold when C is a tight cut.

Theorem 2.6
A matching covered graph G is Pfaffian if and only if each of its bricks and braces is Pfaffian.

This result reduces the scope of problems 1.1 and 1.2 to bricks and braces.

3 Removable Classes

Let G be a matching covered graph. An edge e of G is a *removable edge* if $G - e$ is matching covered. A pair $\{e, f\}$ of edges of G is a *removable doubleton* if $G - e - f$ is matching covered but neither e nor f is removable in G. We shall use the common name *removable class* to designate either a removable edge or a removable doubleton. The following result was proved in [4] (Theorem 5.1).

Theorem 3.1
Let R be a removable class of a matching covered graph G. Then, $b(G - R) \geq b(G)$ if R is a singleton, and $b(G - R) = b(G) - 1$ if R is a doubleton.

As a special case of the second part of the above theorem, we have the following interesting result due to Lovász [13]:

Theorem 3.2
For every removable doubleton $\{e, f\}$ of a brick G, the graph $G - \{e, f\}$ is a bipartite matching covered graph, e joins two vertices in one part of the bipartition of $G - \{e, f\}$ and f two vertices in the other part.

In view of the above result, we see that the class of bricks with a removable doubleton is precisely the class of near-bipartite graphs.

A removable class R of G is *b-invariant* if one of the following two alternatives holds: either (i) R is a singleton and $b(G - R) = b(G)$, or (ii) R is a doubleton. Clearly, if G is bipartite then every removable edge of G is b-invariant. The following more general result was established in [2, Corollary 6.5].

Theorem 3.3
Every removable edge of a solid matching covered graph is b-invariant.

3.1 Removable ears and conformal subgraphs

An *ear* in a matching covered graph G is a path P of odd length in G such that both ends of P have degree at least three in G, but all the internal vertices of P have degree two in G. For an ear P, the graph $G - P$ is the graph obtained from G by deleting all edges and internal vertices of P, and P is said to be *removable* if $G - P$ is matching covered. A *double ear* in G is a pair $\{P_1, P_2\}$ of vertex-disjoint ears. A double ear $\{P_1, P_2\}$ is *removable* if neither P_1 nor P_2 is removable, but $G - P_1 - P_2$ is matching covered. The following theorem is one of the basic results of the theory of matching covered graphs, see Lovász and Plummer[14, Chapter 5].

Theorem 3.4
Let G be a matching covered graph and let H be a matching covered subgraph of G. Then, H is a conformal matching covered subgraph of G if and only if there exists a sequence $(G_1, G_2, \ldots, G_r)$ of subgraphs of G such that $G_1 = G$, $G_r = H$ and, for $2 \leq i \leq r$, G_i is obtained from G_{i-1} by deleting either a removable ear or a removable double ear of G_{i-1}.

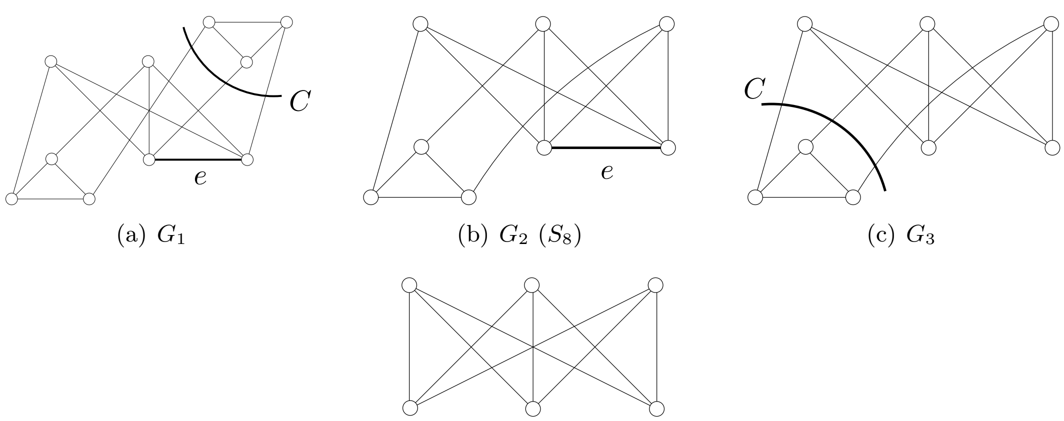
4 Minimal Non-Pfaffian Graphs

4.1 Minors

Let G be a matching covered graph, and let v be a vertex of degree two in G, with neighbours v_1 and v_2. Then $C := \partial(X)$, where $X := \{v, v_1, v_2\}$, is a tight cut of G, and the C-contraction G/X is said to be obtained by *bi-contracting* v from G. (Equivalently, the *bi-contraction* of v from G consists of contracting the two edges incident with v.) Norine and Thomas [18] call a graph H a *matching minor* of a graph G if H can be obtained from a conformal subgraph of G by bi-contractions. A matching minor of G can be obtained from G by deletions of removable classes and bi-contractions. (This follows from Theorem 3.4.) We introduce here the notion of a minor which is stronger than the notion of a matching minor. Norine and Thomas [18] have another notion of a minor, which they do not restrict to matching covered graphs. However, if restricted to matching covered graphs, it is equivalent to the definition of minor given below [16]. In fact, they have discovered an infinite family of non-Pfaffian minimal graphs.

A *deletion-contraction minor* of a matching covered graph G, or simply a *minor* of G, is a graph that is obtainable from G, up to isomorphism, by means of deletions of removable classes and contractions of shores of separating cuts. In other words, H is a minor of G if there exists a sequence $(G_1, G_2, \ldots, G_r)$ of graphs such that, $G_1 = G$, $G_r \cong H$ and, for $1 \leq i \leq r-1$, the graph G_{i+1} is obtained from G_i by either deleting a removable class or by contracting a shore of a separating cut to a single vertex. As an example, consider the sequence (G_1, G_2, G_3, G_4) of graphs in Figure 6. The graph G_1 is a nonsolid brick, the cut C is a (nontight) separating cut of G_1. The graph G_2 is a C-contraction of G_1, a brick (which happens to be a solid brick, denoted in [7] by S_8), and e is a removable edge in it. The cut C in $G_3 := G_2 - e$ is a separating cut (in fact, a tight cut) and G_4 is obtained from G_3 by contracting one of the shores of C. Thus G_1, G_2, G_3 and G_4 are minors of G_1. We allow $r = 1$, that is, we consider that every graph is a minor of itself.

It follows from Theorem 3.4 that every conformal subgraph of a matching covered graph G is a minor of G. Consequently, every matching minor of G is also a minor of G. But not every minor of G is a matching minor of G. For example, $K_{3,3}$ is a minor of G_3, but it is not a matching minor of that graph.



(a) G_1 (b) G_2 (S_8) (c) G_3

(d) G_4

Figure 6: Minors

From Propositions 1.6 and 2.5, we may now deduce the following important property of Pfaffian graphs.

Theorem 4.1
A matching covered graph is Pfaffian if and only if all its minors are Pfaffian.

In light of the above theorem, to show that a given graph is non-Pfaffian, it suffices to produce a minor of that graph which is known to be non-Pfaffian. For example, since $K_{3,3}$ is known to be non-Pfaffian, and since it is a minor of each of the graphs in Figure 6, we may conclude that each of those graphs is non-Pfaffian.

Motivated by the above observation, we define a non-Pfaffian matching covered graph to be *deletion-contraction minimal*, or simply *minimal*, if all its proper minors are Pfaffian. It follows from Theorem 2.6 that every minimal non-Pfaffian matching covered graph is either a brick or a brace. In addition, it is easy to see that such a graph is also simple.

Using the notion of minimal graphs, Little's theorem [11] may now be stated as follows:

Theorem 4.2
The only minimal non-Pfaffian bipartite matching covered graph is the brace $K_{3,3}$.

Although a minimal non-Pfaffian graph cannot contain nontrivial tight cuts, it may contain non-trivial separating cuts. (For example, the Petersen graph has a non-trivial separating cut, but it is a minimal non-Pfaffian brick.) In fact, we shall prove the following surprising general theorem.

Theorem 4.3 (The Main Theorem)
Every minimal non-Pfaffian brick must have a nontrivial separating cut.

Thus, no minimal non-Pfaffian brick can be solid. By Theorem 2.4, it now follows that every minimal non-Pfaffian brick G contains two disjoint odd cycles C_1 and C_2 such that $G - V(C_1 \cup C_2)$ has a perfect matching.

All bricks and braces of a solid matching covered graph are also solid. Since, by the above theorem, there do not exist minimal non-Pfaffian solid bricks, every minimal non-Pfaffian solid matching covered graph is a brace. Little's Theorem 4.2 now implies the following assertion.

Corollary 4.4
The only minimal non-Pfaffian solid matching covered graph is the brace $K_{3,3}$.

Let H be a matching covered graph, and let e be an edge of H. A *bi-subdivision* of e consists of subdividing it by inserting an even number of vertices between its ends. A *bi-subdivision* of H consists of bi-subdividing each of the edges in a subset of $E(H)$. Clearly, if H is matching covered, then any bi-subdivision of H is also matching covered. A graph H is a *conformal minor* of a graph G if some bi-subdivision of H is a conformal subgraph of G. Note that every conformal minor of G is also a matching minor of G.

All previously known characterizations of classes of Pfaffian graphs have been in terms of excluded conformal minors. It is also possible to present a similar characterization of solid matching covered graphs. The following result may be deduced from our Main Theorem 4.3. (We have chosen to not include the proof of this theorem in this paper to limit its length.)

THEOREM 4.5
A solid matching covered graph is Pfaffian if and only if it does not contain $K_{3,3}$ and none of the three graphs shown in Figure 7 as a conformal minor.

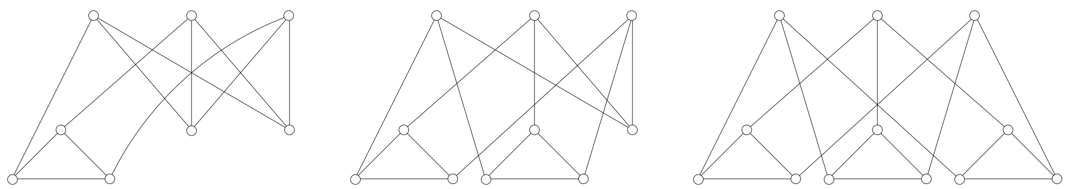


Figure 7: Three graphs obtained by splicing K_4's with $K_{3,3}$

5 e-Triples

This section is dedicated to establishing a number of basic properties of minimal non-Pfaffian graphs. Here we only use the fact a minimal non-Pfaffian matching covered graph does not contain a proper conformal subgraph that is non-Pfaffian.

LEMMA 5.1
Let G be a minimal non-Pfaffian matching covered graph, e a removable edge of G. Then, G has an orientation D and two perfect matchings, M_1 and M_2, such that (i) $D - e$ is a Pfaffian orientation of $G - e$, (ii) edge e lies in $M_1 \cap M_2$, and (iii) M_1 and M_2 have distinct signs in D.

Proof: As G is minimal non-Pfaffian, it is non-Pfaffian, but $G-e$ is Pfaffian. Extend any Pfaffian orientation of $G - e$ to an orientation D of G, by assigning to e an arbitrary orientation. As $D-e$ is a Pfaffian orientation of $G-e$, all perfect matchings of $G-e$ have the same sign, say positive, in D. If all the perfect matchings containing e also have positive sign in D, then D itself would be a Pfaffian orientation of G. And, if all the perfect matchings containing e have negative sign in D, the digraph D' obtained from D by reversing the orientation of the edge e would be a Pfaffian orientation of G. Both these cases are impossible because, by hypothesis, G is non-Pfaffian. We conclude that D must have two perfect matchings M_1 and M_2, both containing e, and having distinct signs. □

The orientation D of G and the perfect matchings M_1 and M_2 of G constitute, in that order, an *e-triple*.

In [6, Corollary 3.6] we observed the following important connection between removable classes and Pfaffian orientations.

Theorem 5.2
Let G be a matching covered graph, R be a b-invariant class of G, and $\overrightarrow{G-R}$ be any Pfaffian orientation of $G-R$. Then G is Pfaffian if and only if there is an extension $\vec{G}$ of $\overrightarrow{G-R}$ which is a Pfaffian orientation of G.

The following corollary of the above theorem will play a pivotal role in the proof of the Main Theorem.

Corollary 5.3
Let G be an minimal non-Pfaffian matching covered graph, e a b-invariant edge of G, and let D an orientation of G, M_1 and M_2 perfect matchings of G such that (D, M_1, M_2) is an e-triple. Then, every b-invariant edge of $G-e$ lies in $M_1 \cup M_2$.

Proof: Assume, to the contrary, that $G-e$ has a b-invariant edge f that does not lie in $M_1 \cup M_2$. We first assert that f is b-invariant in G, and that e is b-invariant in $G-f$. To see this, note first that edge f is removable in $G-e$, therefore $G-e-f$ is matching covered. As M_1 does not contain f, it is a perfect matching of $G-f$. On the other hand, e lies in M_1. Thus, e is admissible in $G-f$. We conclude that $G-f$ is matching covered. As e is b-invariant in G and f is b-invariant in $G-e$, it follows, by the monotonicity of function b (see Theorem 3.1), that

$$b(G) = b(G-e) = b(G-e-f) \geq b(G-f) \geq b(G),$$

whence equality holds throughout. As asserted, f is b-invariant in G and e is b-invariant in $G-f$.

Since $D-e$ is a Pfaffian orientation of $G-e$, it follows that $D-e-f$ is a Pfaffian orientation of $G-e-f$. Also, since G is minimal non-Pfaffian, the graph $G-f$ is Pfaffian. Thus, as e is b-invariant in $G-f$, it follows from Theorem 5.2 that $D-e-f$ has an extension to a Pfaffian orientation, say D', of $G-f$. As f does not lie in $M_1 \cup M_2$ (by our assumption), M_1 and M_2 are perfect matchings of $G-f$. By hypothesis, (D, M_1, M_2) is an e-triple. Thus M_1 and M_2 have distinct signs in D, and they would have distinct signs in D', regardless of the direction assigned to f. This is impossible because D' is Pfaffian. Hence, as asserted, every b-invariant edge of $G-e$ lies in $M_1 \cup M_2$. □

Lemma 5.4
Let G be a minimal non-Pfaffian brick, e a b-invariant edge of G, and let D an orientation of G, M_1 and M_2 perfect matchings of G such that (D, M_1, M_2) is an e-triple. Then, every removable doubleton of $G-e$ is a subset of $M_1 \cup M_2$.

Proof: Assume, to the contrary, that $G-e$ has a removable doubleton $R := \{f_1, f_2\}$ that is not a subset of $M_1 \cup M_2$. Adjust notation so that f_2 does not lie in $M_1 \cup M_2$. By hypothesis, G is a brick, e is a b-invariant edge of G and R is a removable doubleton of $G-e$. Thus,

$$b(G-e) = b(G) = 1 \quad \text{and} \quad b(G-e-R) = 0.$$

We deduce that $G - e - R$ is bipartite. Let $\{A_1, A_2\}$ denote the bipartition of $G - e - R$. Then, one of f_1 and f_2 has both ends in A_1, the other has both ends in A_2. Adjust notation so that f_i has both ends in A_i, for $i = 1, 2$.

Let $n = |M_1| = |M_2|$. For $i = 1, 2$, let m_i denote the number of edges of $M_i - e$ that join a vertex of A_1 to a vertex of A_2. Clearly, $m_i < n$. Let x_i denote the number of ends of e in A_i. Edge f_2 is the only edge of $G - e$ having both ends in A_2. Moreover, f_2 does not lie in $M_1 \cup M_2$, by hypothesis. Thus, $m_1 = m_2 = n - x_2$. Consequently, $x_2 \geq 1$. That is, edge e has at least one end in A_2.

Consider first the case in which edge e has both ends in A_2. Then, $m_i = n-2$. In that case, f_1 lies in $M_1 \cap M_2$. The graph $G-e-R$, that is, $G-e-f_1-f_2$, is matching covered and bipartite. The set M_i is a perfect matching of $G - f_2$. Thus, f_2 is removable in G. Moreover, neither e nor f_1 is removable in $G-f_2$. Thus, $S := \{e, f_1\}$ is a removable doubleton of $G-f_2$. We deduce that $b(G-f_2)-1 = b(G-f_2-S) = 0$, whence f_2 is b-invariant in G. This is a contradiction to Corollary 5.3, as f_2 does not lie in $M_1 \cup M_2$.

Consider next the case in which edge e has one end in A_1, the other end in A_2. Then, $m_i = n - 1$. Consequently, f_1 does not lie in M_i. Edge e is thus admissible in $G - R$. Moreover, $G - R$ is bipartite. We deduce that e is b-invariant in $G - R$. By the definition of an e-triple, $D - e$ is a Pfaffian orientation of $G - e$. It follows that $D - e - R$ is a Pfaffian orientation of $G - e - R$. As G is minimal non-Pfaffian, the graph $G - R$ is Pfaffian. Thus, as e is b-invariant in $G - R$, by Theorem 5.2, $D - e - R$ has an extension to a Pfaffian orientation, say D', of $G - R$. But, as R is disjoint from $M_1 \cup M_2$, it follows that M_1 and M_2 are perfect matchings of $G - R$. By hypothesis, (D, M_1, M_2) is an e-triple. Thus M_1 and M_2 have distinct signs in D, and they would have distinct signs in D', regardless of the directions assigned to f_1 and f_2. This is impossible because D' is Pfaffian.

In both alternatives considered, we derived a contradiction. As asserted, R is a subset of $M_1 \cup M_2$. □

6 Admissible and Removable Edges

It should be clear from the foregoing discussion that one possible way of showing that a non-Pfaffian matching covered graph is not minimal is by showing that it has removable classes satisfying suitable properties. By using results concerning non-removable edges in bipartite graphs, we were able to present a simple proof of Theorem 4.2 in [6]. The proof of the Main Theorem 4.3 requires a deeper understanding of non-removable edges in matching covered graphs, especially in solid bricks. We develop these results in this section.

6.1 The three case lemma

Let G be a matching covered graph, let e be a removable edge of G. Let $C := \partial(X)$ be a cut of G. We say that C is *peripheral* if C is nontrivial, cut $C - e$ is tight in

$G-e$ and a $(C-e)$-contraction is bipartite. Assume that C is peripheral, where $J := (G-e)/\overline{X} \to \overline{x}$ has bipartition $\{B, I\}$, with $\overline{x}$ in I. We then refer to $I - \overline{x}$ as the *inner part* of J, whereas B is the *outer part* of J. The following property is easily proved:

Lemma 6.1
Let G be a matching covered graph free of nontrivial tight cuts, let e be a b-invariant edge of G, $C := \partial(X)$ a nontrivial cut of G such that $C-e$ is tight in $G-e$. Then, C is peripheral and precisely one of $(C-e)$-contractions of $G-e$ is bipartite. Let $J := (G-e)/\overline{X} \to \overline{x}$ be bipartite. Then, either both ends of e lie in the inner part of J or one end of e lies in the inner part of J, the other end lies in $\overline{X}$. □

Lemma 6.2 (The Three Case Lemma)
Let G be a brick, e a b-invariant edge of G such that $G-e$ is not a brick. Let H be the brick of $G-e$, obtained by a tight cut decomposition of $G-e$. Then, one of the following three alternatives holds (see Figure 8):

(i) *either G has a peripheral cut $C_1 := \partial(X_1)$ such that $J_1 := (G-e)/\overline{X_1} \to \overline{x_1}$ is bipartite, $H = (G-e)/X_1 \to x_1$ and edge e has one end in the inner part of J_1, the other end in $V(H) - x_1$,*

(ii) *or G has two peripheral cuts $C_i := \partial(X_i)$, for $i = 1, 2$, such that X_1 and X_2 are disjoint, $J_i := (G-e)/\overline{X_i} \to \overline{x_i}$ is bipartite, $H = ((G-e)/X_1 \to x_1)/X_2 \to x_2$ and edge e has one end in the inner part of J_1, the other end in the inner part of J_2,*

(iii) *or G has a peripheral cut $C_1 := \partial(X_1)$ such that $J_1 := (G-e)/\overline{X_1} \to \overline{x_1}$ is bipartite, $H = (G-e)/X_1 \to x_1$ and edge e has both ends in the inner part of J_1.*

Proof: Assume that $G-e$ is not a brick. Let $C_1 := \partial(X_1)$ be a nontrivial cut of G such that $C_1 - e$ is tight and one of the contraction vertices of H is x_1, obtained by contracting X_1 to x_1. As e is b-invariant, J_1 is bipartite. Moreover, either e has both ends in the inner part of J_1, or e has one end in the inner part of J_1, the other end in $\overline{X_1}$. In the former case, we have the last of the three asserted cases. Assume thus that e has one end in the inner part of J_1, the other end in $\overline{X}$. If x_1 is the only contraction vertex of H then the first of the three cases holds. We may thus assume that H has more than one contraction vertex. Let x_2 be another contraction vertex of H, distinct from x_1. Let $J_2 := (G-e)/\overline{X_2} \to \overline{x_2}$. Then, J_2 is bipartite. Moreover, edge e has one end in the inner part of J_2. This conclusion holds for each contraction vertex x_2 of H distinct from x_1. We deduce that H has precisely two contraction vertices. Moreover, the second of the three cases holds. □

Let G be a brick, e a b-invariant edge of G. If $G-e$ is also a brick then we say that e has *index* zero. If $G-e$ is not a brick, then we way that e has *index* one, two or three, depending on which of the three cases stated in Lemma 6.2 holds. If H has

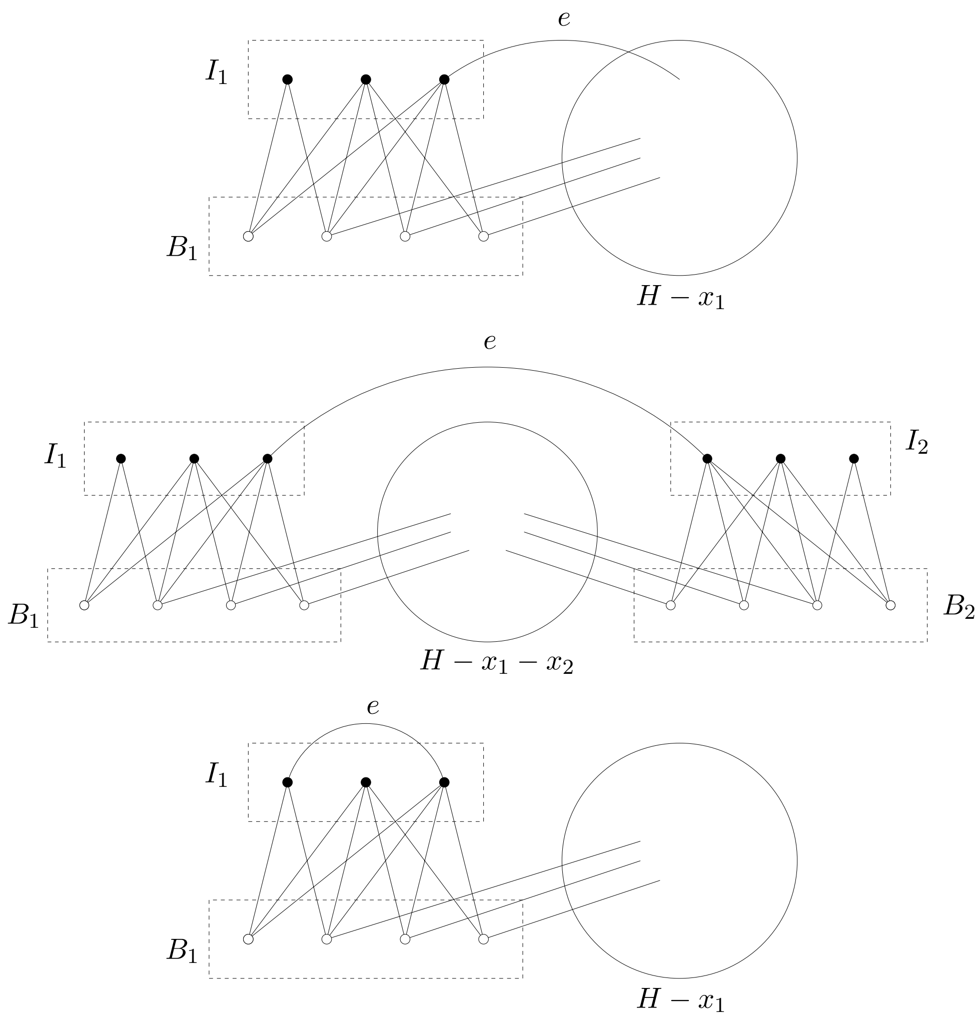


Figure 8: The three cases of Lemma 6.2

one contraction vertex and one of the ends of e lies in $V(H)$ then e has index one. If H has two contraction vertices then e has index two. Finally, if the last of the three cases holds then e has index three.

6.2 Thin edges

Recall that the bi-contraction of a vertex of degree two in a graph consists of contracting both the edges incident with that vertex. If G is a brick, and e is an edge of G, then $G - e$ has at most two vertices of degree two. The *retract* of $G - e$ is the graph obtained from it by bi-contracting all its vertices of degree two. An edge e of a brick G is *thin* if the retract of $G - e$ is a brick. (Thus thin edges of bricks are special types of b-invariant edges.) We remark that the index of a thin edge e of a brick G is

- *zero*, if both ends of e have degree four or more in G;
- *one*, if exactly one end of e has degree three in G;
- *two*, if both ends of e have degree three in G and edge e does not lie in a triangle;
- *three*, if both ends of e have degree three in G and edge e lies in a triangle.

Examples of thin edges of indices one, two, and three are indicated by solid lines in the three bricks, respectively, shown in Figure 9.

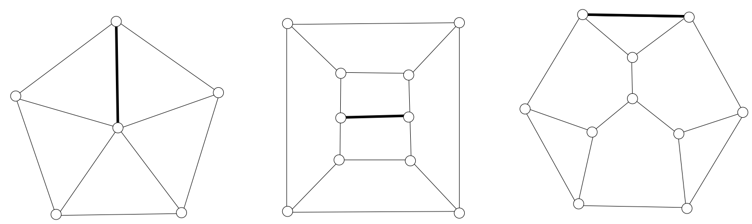


Figure 9: Thin edges of indices one two and three

In [7], we proved the existence of thin edges for bricks. (See also Norine and Thomas [18].)

THEOREM 6.3 (THE THIN EDGE THEOREM FOR BRICKS)
Every brick distinct from K_4, $\overline{C_6}$ and the Petersen graph has a thin edge. □

6.3 Removable edges in bipartite graphs

The following result provides a characterization of non-removable edges in bipartite matching covered graphs.

PROPOSITION 6.4 (SEE [14])
Let G be a bipartite matching covered graph with bipartition (A, B), and let e be an edge of G. Then, e is not removable in G if and only if there is a partition (A', A'') of A and a partition (B', B'') of B with $|A'| = |B'|$ such that e is the only edge joining a vertex in A' to a vertex in B''.

We shall now establish a general result concerning bricks and peripheral cuts which provides an essential tool in achieving our objective.

LEMMA 6.5
Let G be a matching covered graph free of nontrivial tight cuts, e a removable edge of G, $C := \partial(X)$ a peripheral cut of G such that the $(C - e)$-contraction $J := (G - e)/\overline{X} \to \overline{x}$ is bipartite. The following properties hold: (i) every edge of $C - e$ is removable in J and (ii) for any inner vertex v of J having degree three or more, at most one edge of J in $\partial(v)$ is not removable.

<u>Proof</u>: Let B and I denote the outer and inner parts of J, respectively. By Lemma 6.1, edge e has one end in I, and no end in B.

Let f be a non-removable edge of J. Let v be the end of f in I, w its end in B. Let u be the end of f in G distinct from w. Thus, either (i) u lies in $\overline{X}$ and $v = \overline{x}$ or (ii) $u = v$ and v lies in I. By Proposition 6.4, there exists a partition (B', B'') of B and a partition (I', I'') of $I + \overline{x}$ such that $|I'| = |B'|$ and f is the only edge of J that joins a vertex of I' to a vertex of B'' (see Figure 10). If the contraction

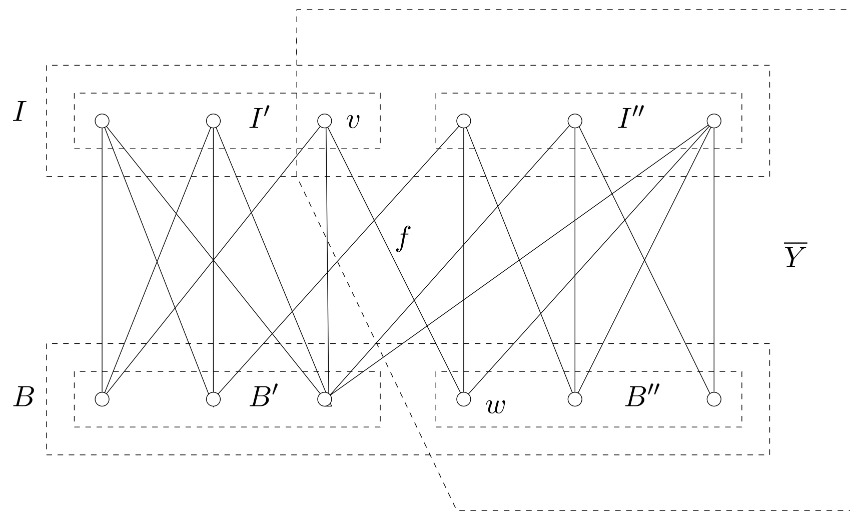


Figure 10: An illustration for the proof of Lemma 6.5

vertex $\overline{x}$ does not lie in I'' then the set $I'' \cup \{u\}$ is a nontrivial barrier of G, and the cut $\partial(I'' \cup B'' \cup \{u\})$ a nontrivial tight cut of G, a contradiction. We deduce that the contraction vertex $\overline{x}$ lies in I''. Edge f has no end in I''. In particular, f is not incident with $\overline{x}$. That is, f does not lie in $C - e$. This conclusion holds for each non-removable edge f of J. Thus, every edge of $C - e$ is removable in J. This concludes the first part of the proof.

To prove the second part, assume that v lies in I and that the degree of v in J is at least three. Observe that B' is a (possibly trivial) barrier of $G - e$. Let $Y := B' \cup (I' - v)$, let $D := \partial(Y)$. Then, $D - e$ is a tight cut of $G - e$. If $I' = \{v\}$ then all the edges of $J - f$ incident with v are multiple edges of J, because the degree of v in J is three or more: in that case, all the edges of $\partial_J(v) - f$ are removable in J. We may thus assume that $I' \neq \{v\}$. Then, D is nontrivial. By the first part, every edge of $D - e$ is removable in the $(D - e)$-contraction $K := (G - e)/\overline{Y} \to \overline{y}$ of $G - e$. In particular, every edge of $\partial_K(v) - f$ is removable in K. Note that K is a $D - e$-contraction of J. The edges of $\partial(v) - f$ are multiple edges in the $(D - e)$-contraction $(J)/Y \to y$ of J, because the degree of v is three or more in J. Thus, every edge of $\partial_J(v) - f$ is removable in both $(D - e)$-contractions of J. As asserted, at most one edge of J incident with v is not removable in J. $\square$

Corollary 6.6
Let G be a graph free of nontrivial tight cuts, e a removable edge of G, $C := \partial(X)$ a peripheral cut of G, let $J := (G-e)/\overline{X} \to \overline{x}$ be a bipartite $(C-e)$-contraction of $G-e$. Then, every removable class of $(G-e)/X \to x$ is also removable in $G-e$. □

6.4 Barriers and non-removable edges

There is a close connection between removable edges in a matching covered graph and its barriers. In this section we examine this relationship and establish a number of basic results.

Let G be a (not necessarily matching covered) graph. We say that G is *even* if $|V(G)|$ is even and *odd* if $|V(G)|$ is odd. We denote by $\mathcal{O}(G)$ the set of odd components of G. Using this notation, Tutte's fundamental theorem may be stated as follows:

Theorem 6.7 (Tutte's Perfect Matching Theorem [20])
A graph G has a perfect matching if and only if $|\mathcal{O}(G-S)| \leq |S|$, for each subset S of $V(G)$.

A nonempty set B of vertices of a graph G is a *barrier* of G if $|\mathcal{O}(G-B)| = |B|$. If G is matching covered then, for every $v \in V$, the set $\{v\}$ is a barrier. Such barriers are *trivial*.

Recall that a graph G is *critical* (or, *factor-critical*) if, for any vertex v of G, the subgraph $G-v$ has a perfect matching. The following corollary of Tutte's theorem may be derived using standard techniques of matching theory.

Corollary 6.8
Let G be a graph which has a perfect matching. Then the following properties hold:

(i) *An edge e of G is admissible if, and only if, there is no barrier which contains both ends of e.*

(ii) *For each maximal barrier B of G, all components of $G-B$ are critical.*

A graph G is *bicritical* if, for any two distinct vertices v and w of G, subgraph $G-v-w$ has a perfect matching. By Tutte's Perfect Matching Theorem, it is easy to see that a graph with an even number of vertices is bicritical if and only if it has only trivial barriers. Edmonds, Lovász and Pulleyblank [8] proved that a matching covered graph on four or more vertices is a brick if and only if it is 3-connected and bicritical.

If an edge e of a matching covered graph G is not removable, by definition, there must be inadmissible edges in $G-e$. Thus, by Corollary 6.8, $G-e$ must necessarily contain a barrier which includes both ends of some edge. This observation is the basis of all the known criteria for deciding whether or not a given edge of a matching covered graph is admissible.

As we shall see, the above proposition plays a very useful role in deriving results concerning removable edges in matching covered graphs. A similar result for non-bipartite matching covered graphs would be desirable, but none is known. However, we have been able to find a useful theorem concerning non-removable edges in bicritical graphs. Its proof requires the following classical result:

Theorem 6.9 (Dulmage-Mendelson Decomposition Theorem [14])
Let G be a graph with a perfect matching and bipartition (A, B). Then, there exists a partition $(A_1, A_2, \ldots, A_r)$ of A and a partition $(B_1, B_2, \ldots, B_r)$ of B, where $r \geq 1$ and such that for $i = 1, \ldots, r$, (i) the subgraph G_i of G induced by $A_i \cup B_i$ has bipartition (A_i, B_i) and is matching covered and (ii) every edge of G incident with some vertex of A_i is also incident with some vertex of B_j, where $j \leq i$.

We now present that structural theorem concerning non-removable edges in bicritical graphs to which we alluded to earlier. For a barrier B of a graph G, the *bipartite graph $H(B)$ associated with B* is obtained by deleting all edges with both ends in B, deleting all vertices in the even components of $G-B$, and then contracting each odd component of $G - B$ to a single vertex.

Theorem 6.10
Let G be a bicritical graph, and let e be a non-removable edge of G. Then, $G - e$ contains a barrier B that satisfies the following properties:

(i) *Bipartite graph $H(B)$ associated with barrier B is matching covered.*

(ii) *Edge e has its ends in distinct odd components of $G - e - B$.*

(iii) *For each odd component K of $G-e-B$, cut $C(K) := \partial_G(V(K))$ is separating in G.*

Proof: As e is non-removable, graph $G - e$ is not matching covered. By Corollary 6.8, $G - e$ has a barrier that contains both ends of some edge f. Let $B^\star$ denote a maximal barrier of $G - e$ that contains both ends of f. By the maximality of $B^\star$, it follows from part (ii) of Corollary 6.8 that $G - e - B^\star$ contains only odd components. Moreover, each component of $G - e - B^\star$ is critical.

As f is admissible in G, it follows that e has its ends in distinct (odd) components of $G - e - B^\star$. Consider now the bipartite graph $H(B^\star)$ associated with $B^\star$. Let M be any perfect matching of G that does not contain edge e. Then, for each (odd) component K of $G - e - B^\star$, M contains precisely one edge in cut $\partial_G(V(K))$. It follows that the restriction of M to $H(B^\star)$ is a perfect matching of $H(B^\star)$. Let $A^\star$ denote the part of the bipartition of $H(B^\star)$ distinct from $B^\star$. By the Dulmage-Mendelsohn Decomposition Theorem, there exists a partition $(A_1, A_2, \ldots, A_r)$ of $A^\star$ and a partition $(B_1, B_2, \ldots, B_r)$ of $B^\star$, where $r \geq 1$ and such that for $i = 1, 2, \ldots, r$, (i) the subgraph H_i of $H(B^\star)$ induced by $A_i \cup B_i$ has bipartition (A_i, B_i) and is matching covered and (ii) every edge of $H(B^\star)$ incident with some vertex of A_i is also incident with some vertex of B_j, where $j \leq i$. See Figure 11.

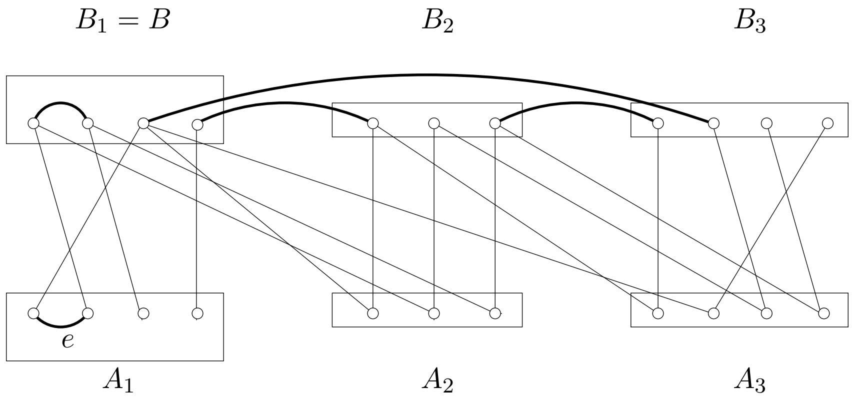


Figure 11: The graph L in the proof of Theorem 6.10

Let $B := B_1$. Then, B is a barrier of $G - e$. Moreover, $H(B)$ is equal to H_1, whence $H(B)$ is matching covered. This proves part (i).

If edge e has at least one end, say v, in some even component of $G - e - B$, then $B + v$ would be a nontrivial barrier of G, a contradiction to the hypothesis that G is bicritical. Thus, e has both ends in odd components of $G - e - B$. Every odd component of $G - e - B$ is a component of $G - e - B^\star$. Moreover, edge e has its ends in distinct components of $G - e - B^\star$. Thus, edge e has its ends in distinct odd components of $G - e - B$. This proves part (ii).

Every odd component of $G - e - B$ is critical because it is also a component of $G - e - B^\star$. Thus, for each odd component K of $G - e - B$, graph $G(K) := G/\overline{V(K)} \to \overline{v_K}$ is matching covered, in fact, bicritical. (The vertex set of $G(K)$ is $V(K) + \overline{v_K}$. Since K is critical, $\overline{v_K}$ is not contained in any nontrivial barrier of $G(K)$. Furthermore, any subset of $V(K)$ that is a barrier of $G(K)$ would also be a barrier of G. Since G is bicritical, it follows that $G(K)$ is bicritical.)

Now let L be the graph obtained from G by contracting the vertex set of each odd component of $G - e - B$ to a single vertex. In order to prove that cut $C(K)$ is separating in G, for each odd component K of $G - e - B$, it suffices to show that graph L is matching covered.

For i, $1 \leq i \leq r$, let L_i denote the subgraph of L induced by $B_i \cup A_i$. We first note that if g is an edge of L that is not an edge of any L_i, any perfect matching M of G containing g must contain an edge with both ends in $B^\star$. This is clearly true if $g = e$ or g itself has both ends in $B^\star$. If not, g has one end in some B_j and one end in some A_i, where $j < i$. If a perfect matching M containing g contains no edges of G with both ends in $B^\star$, then it must match each vertex of $B^\star$ with precisely one odd component of $G - B^\star$. This is not possible because all neighbours of the vertices in $B_i \cup B_{i+1} \cup \cdots \cup B_r$ are in $A_i \cup A_{i+1} \cup \cdots \cup A_r$, and the edge g has one end in B_j, $(j < i)$, and one end in A_i.

We now proceed to show that every edge of L is admissible. By the definition of B_i and A_i, each L_i is matching covered. Furthermore, if for i, $1 \leq i \leq r$, M_i is any perfect matching of L_i, then $M_1 \cup M_2 \cup \cdots \cup M_r$ is a perfect matching of L. It follows that every edge of L that is an edge of some subgraph L_i is admissible in L. Thus, let g be an edge of L that is not an edge of any L_i. By the above observation, there is a perfect matching M of G containing g and some edge h of G which has both its ends in $B^\star$. A simple counting argument shows that M must contain e and, for each odd component K of $G - B^\star$, exactly one edge in $\partial(V(K))$. The restriction N of M to $E(L)$ is then a perfect matching of L. This establishes the validity of part (iii). □

We conclude this subsection with a simple lemma which will be used in the proof of Theorem 6.13 concerning non-removable edges in solid bricks.

Lemma 6.11
Let G be a brick, and let e_1 and e_2 be two adjacent non-removable edges of G. Suppose that, for $i = 1, 2$, B'_i is a barrier of $G - e_i$. Then $|B'_1 \cap B'_2| \leq 1$.

Proof: Assume, to the contrary, that B'_1 and B'_2 have two vertices, say x and y, in common. As G is a brick $G - x - y$ has a perfect matching, say M. However, as B'_i is a barrier of $G - e_i$, for $i = 1, 2$, it follows that both e_1 and e_2 belong to M. This is impossible because e_1 and e_2 are adjacent edges. □

6.5 Removable edges in solid bricks

We shall now use Theorem 6.10 to derive useful results concerning removable edges in solid bricks. We begin with a simple consequence of that theorem.

Corollary 6.12
Let G be a solid brick, e a non-removable edge of G. Then, $G - e$ contains a barrier B such that every odd component of $G - e - B$ is trivial and edge e has its ends in distinct odd components of $G - e - B$. Moreover, the graph $H(B)$ obtained from $G - e$ by removing the vertices in even components of $G - e - B$ and the edges having both ends in B is matching covered.

Proof: Let B denote a barrier that satisfies the statement of Theorem 6.10. Edge e has its ends in distinct odd components of $G - e - B$. Thus, B is nontrivial. Let K be any odd component of $G - e - B$. Then, $C(K) := \partial(V(K))$ is separating in G. By hypothesis, G is solid. Therefore, $C(K)$ is tight. By hypothesis, G is a brick. Therefore, $C(K)$ is trivial. As B is nontrivial, the set $\overline{V(K)}$ contains three or more vertices. We deduce that $V(K)$ is a singleton. This conclusion holds for each odd component K of $G - e - B$. The assertion now follows, by the properties of $H(B)$ in the statement of Theorem 6.10. □

Theorem 6.13
Let G be a solid brick, v a vertex of G, n the number of neighbours of v, d the degree of v. Enumerate the d edges of $\partial(v)$ as $e_i := vv_i$, for $i = 1, 2, \ldots, d$. Assume

that neither e_1 nor e_2 is removable in G. Then, $n = 3$ and, for $i = 1, 2$, there exists an equipartition (B_i, I_i) of $V(G)$ such that

(i) *e_i is the only edge of G that has both ends in I_i,*

(ii) *every edge that has both ends in B_i is incident with v_3, and*

(iii) *the bipartite subgraph H_i of G, obtained by the removal of e_i and each edge having both ends in B_i, is matching covered.*

Moreover, $B_1 = (I_2 - v) \cup \{v_3\}$ and $B_2 = (I_1 - v) \cup \{v_3\}$. (See Figure 12 for an illustration.)

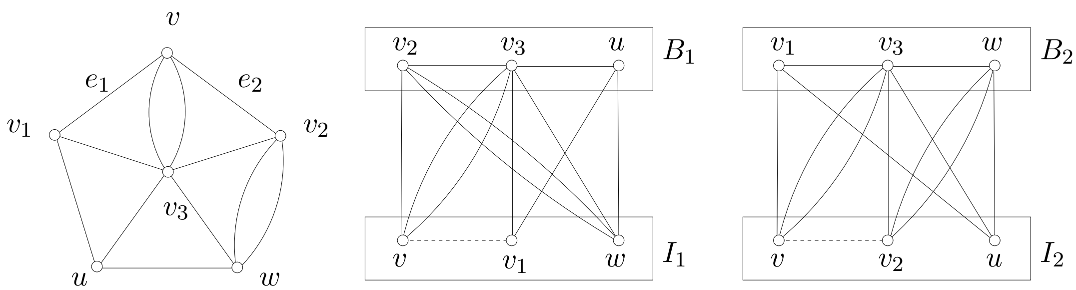


Figure 12: Graphs G, $G - e_1$ and $G - e_2$

Proof: By hypothesis, neither e_1 nor e_2 is removable in G. By Corollary 6.12, for $i = 1, 2$, graph $G - e_i$ has a barrier B_i such that (i) each odd component of $G - e_i - B_i$ is trivial, (ii) edge e_i has its ends in distinct odd components of $G - e_i - B_i$, and (iii) the bipartite graph H_i associated with B_i is matching covered.

Let us now prove that $n = 3$ and $B_1 \cap B_2 = \{v_3\}$. As e_i has both ends in odd components of $G - e_i - B_i$, and since each such odd component is trivial, it follows that v is the vertex of a trivial component of $G - e_i - B_i$. No edge of $\partial(v) - e_i$ has both ends in distinct odd components of $G - e_i - B_i$, because B_i is a barrier of $G - e_i$. It follows that $\{v_3, v_4, \ldots, v_d\}$ is a subset of $B_1 \cap B_2$. By Lemma 6.11, $B_1 \cap B_2$ is either empty or a singleton. We deduce that $n = 3$ and $B_1 \cap B_2 = \{v_3\}$. Consequently, if $d > 3$ then edges $e_3, \ldots, e_d$ are multiple edges.

For $i = 1, 2$, if x is a vertex in an even component of $G - e_i - B_i$, $B_i + x$ is a barrier of $G - e_i$. Using this fact, we now proceed to show that $G - e_i - B_i$ has no even components.

Suppose that $G - e_1 - B_1$ has even components and that xy is an edge of such a component. If x is in an even component of $G - e_2 - B_2$, then $B_1 + x$ and $B_2 + x$ are barriers of $G - e_1$ and $G - e_2$, respectively, and $(B_1 + x) \cap (B_2 + x) \supseteq \{v_3, x\}$. Similarly, if x is in B_2, $B_1 + x$ and B_2 are barriers of $G - e_1$ and $G - e_2$ and $(B_1 + x) \cap B_2 \supseteq \{v_3, x\}$. In either case, we have a contradiction by Lemma 6.11. It follows that x and, similarly, y are isolated vertices of $G - e_2 - B_2$. This is absurd because xy is an edge of $G - e_2 - B_2$ different from e_1. Hence $G - e_1 - B_1$ and, similarly, $G - e_2 - B_2$ have no even components.

For $i = 1, 2$, now let $I_i := V(G) - B_i$. Then, I_i is the set of isolated vertices of $G - e_i - B_i$ and edge e_i is the only edge of G that has both ends in I_i. Moreover, H_i is matching covered. As $B_1 \cap B_2 = \{v_3\}$, and since v lies in $I_1 \cap I_2$, it follows that $B_1 = (I_2 - v) \cup \{v_3\}$ and $B_2 = (I_1 - v) \cup \{v_3\}$. As e_1 is the only edge of G having both ends in I_1, it follows that every edge having both ends in B_2 is incident with v_3. Likewise, every edge having both ends in B_1 is incident with v_3. □

Corollary 6.14
If G is a solid brick with six vertices or more then for every vertex v of G at most two edges incident with v are non-removable in G.

Proof: Let v be a vertex of G, adopt the notation of the statement of Theorem 6.13. Assume that for $i = 1, 2, 3$, edge e_i is not removable in G. Then, for $i = 1, 2, 3$, $V(G)$ has an equipartition (B_i, I_i) such that $B_1 \cap B_2 = \{v_3\}$, $B_1 = (I_3 - v) \cup \{v_2\}$ and $B_2 = (I_3 - v) \cup \{v_1\}$. Then, $I_3 - v = B_1 \cap B_2 = \{v_3\}$, whence $I_3 = \{v, v_3\}$. In that case, as I_3 contains half the vertices of G, it follows that G contains precisely four vertices. □

Corollary 6.15
If G is a solid brick of maximum degree three or four, then, for every vertex v of G at most one edge incident with v does not lie in a removable class of G.

Proof: Adopt the notation in the statement of Theorem 6.13, assume that neither e_1 nor e_2 lies in a removable class of G. Then, neither e_1 nor e_2 is removable in G. By the same theorem, v has precisely three neighbours. Let B_1 and B_2 be as in the statement of Theorem 6.13. For $i = 1, 2$, let n_i denote the number of edges of G that have both ends in B_i. As $|B_i| = |I_i|$ and since e_i has both ends in I_i, it follows that $n_i > 0$, otherwise e_i would not be admissible in G. As $I_1 - v$ and $I_2 - v$ are stable sets, it follows that all edges with both ends in B_1, and all edges with both ends in B_2, are incident with v_3. By hypothesis, the degree of v_3 is three or four. As v_3 is adjacent to v which is a vertex of $I_1 \cap I_2$, it follows that $n_1 + n_2 \leq 3$, whence at least one of n_1 and n_2 is equal to one. Adjust notation so that $n_1 = 1$. Let f_1 denote the only edge of G having both ends in B_1. Then, $H_1 = G - e_1 - f_1$. Neither e_1 is admissible in $G - f_1$ nor f_1 is admissible in $G - e_1$. We deduce that $\{e_1, f_1\}$ is a removable doubleton of G. This contradicts the assumption that neither e_1 nor e_2 lies in a removable class of G. □

7 Proof of the Main Theorem

The Main Theorem 4.3 is clearly equivalent to the statement that there do not exist minimal non-Pfaffian solid bricks. We shall demonstrate this by assuming such a solid brick exists, analyzing the properties of that hypothetical brick, and arriving at a contradiction.

The following result is an immediate consequence of one our earlier results [2, Theorem 2.28]

LEMMA 7.1
If G is a solid matching covered graph and e is a removable edge of G then $G-e$ is also solid.

Combining this result with Theorem 3.3, we deduce that if G is a solid matching covered graph and e is a removable edge of G then e is b-invariant and $G-e$ is solid. We shall use this property several times in our analysis in this section.

In the rest of this section, G denotes a (hypothetical) minimal non-Pfaffian solid brick.

LEMMA 7.2
Brick G is simple and has eight or more vertices. Moreover, it has a thin edge.

Proof: By minimality, G is simple. Graph K_4 is the only simple brick on four vertices. In [7, Theorem 44], we proved that W_5, the wheel on six vertices, is the only simple solid brick on six vertices. But K_4 and W_5, being planar, are Pfaffian. It follows that G has eight or more vertices. Graph $\overline{C_6}$ and the Petersen graph are not solid. Thus, by Theorem 6.3, G has a thin edge. □

LEMMA 7.3
For every removable edge e of G, the brick H of $G-e$ is solid.

Proof: The set of perfect matchings of H is the restriction to $E(H)$ of the set of perfect matchings of $G-e$. Thus, by Proposition 2.2, every separating cut of H is a separating cut of $G-e$. Moreover, every nontrivial cut of H is a nontrivial cut of $G-e$. As $G-e$ is solid, every nontrivial separating cut is tight. As H, a brick, is free of nontrivial tight cuts, it follows that H is free of nontrivial separating cuts. Thus H is a solid brick. □

LEMMA 7.4
Brick G is cubic.

Proof: Assume, to the contrary, that G has a vertex v of degree four or more. By Corollary 6.14, at least two edges incident with v are removable. Let e be a removable edge of G incident with v. As G is solid, e is b-invariant in G. As G is minimal, let (D, M_1, M_2) be an e-triple. We derive now a contradiction by proving that $\partial(v)-e$ has an edge f that lies in a removable class R of $G-e$ disjoint with $M_1 \cup M_2$. If $R=\{f\}$ then, as $G-e$ is solid, it follows that f is b-invariant in $G-e$; moreover, f does not lie in $M_1 \cup M_2$: this is a contradiction to Corollary 5.3. Alternatively, if R is a removable doubleton, then, as it is disjoint with $M_1 \cup M_2$ we arrive at a contradiction to Lemma 5.4

Consider first the case in which G has a peripheral cut $C := \partial(X)$ such that $J := (G - e)/\overline{X} \to \overline{x}$ is bipartite and v lies in the inner part of J. By Lemma 6.5, at most one edge of J incident with v is not removable in J. Thus, at most one edge of $\partial(v) - e$ is not removable in $G - e$. Let f be any of the edges of $\partial(v) - e$ that is removable in $G - e$. Then, f is a removable edge of $G - e$ that does not lie in $M_1 \cup M_2$.

Consider next the case in which v is a vertex of the brick of $G - e$. If $G - e$ has six or more vertices then, by Corollary 6.14, $\partial(v) - e$ contains an edge f that is removable in $G - e$. We may thus assume that H has precisely four vertices. In that case, $G - e$ is not a brick, because G has eight or more vertices. As v, an end of e, lies in H, it follows that e has index one. Let $C := \partial(X)$ denote the peripheral cut of G such that $H = (G - e)/X \to x$. Thus, x and v are two of the four vertices of H. Let u and w denote the other two vertices. Clearly, $|C \cap M_i| = 3$, for $i = 1, 2$. Thus, each of u, v and w is incident with an edge of M_i. It follows that edge uw does not lie in $M_1 \cup M_2$. If v is joined to x by two or more edges then $\partial(v) - e$ has an edge f removable in H. If v and x are joined by precisely one edge then that edge and uw constitute a removable doubleton of H. In both alternatives, H has a removable class R disjoint with $M_1 \cup M_2$. By Corollary 6.6, R is removable in $G - e$. In all alternatives considered, we derived a contradiction. □

Let $e := v'v''$ be a thin edge of G. By Lemma 7.1, matching covered graph $G - e$ is solid. Also, by Theorem 3.3, $G - e$ is a near-brick. Let H denote the brick of $G - e$. As G is cubic, edge e has index two or three.

Lemma 7.5
Brick G is free of triangles and edge e has index two.

Proof: Assume, to the contrary, that G has a triangle, T. Let $C := \partial(T)$, $G' := G/V(T)$ be the C-contraction of G obtained by contracting T to a single vertex. As G is a brick, G' is 3-edge-connected. As G is cubic, G' is also cubic. By Tutte's Perfect Matching Theorem, G' is matching covered. The other C-contraction of G is K_4, also a matching covered graph. Thus, C is a separating cut of G. As G has eight or more vertices, it follows that C is nontrivial. This is a contradiction, because G is solid and free of nontrivial tight cuts, As asserted, G is free of triangles.

We have seen that the index of e is two or three. But it cannot be three, otherwise G would have a triangle. As asserted, the index of e is equal to two. □

Lemma 7.6
$M_1 \cap M_2 = \{e\}$.

Proof: We know that e lies in $M_1 \cap M_2$. Assume, to the contrary, that $M_1 \cap M_2 - e$ contains an edge, f. Consider first the case in which f joins contraction vertices x_1 and x_2. In G, edge f joins a vertex x' adjacent to v' to a vertex x'' adjacent to v''. Then, $\{v', x', x'', v''\}$ is the vertex set of a quadrilateral Q that contains edges e and f and is M_i-alternating, for $i = 1, 2$. In that case, let $M_i' := M_i \triangle E(Q)$.

Each $M_1'M_2'$-alternating cycle is also an M_1M_2-alternating cycle, and *vice versa*. Thus, the signs of M_1' and M_2' in D are distinct. Moreover, M_1' and M_2' are perfect matchings of $G - e$. This is impossible because $D - e$ is Pfaffian (by the definition of D).

We may thus assume that edge f is incident with a vertex v in $V(H) - x_1 - x_2$. The maximum degree of vertices of H is four. By Corollary 6.15, at most one edge incident with v does not lie in a removable class of H. Thus, H has an edge, g, that is incident with v, does not lie in $M_1 \cup M_2$ and lies in a removable class R of H. By Corollary 6.6, R is a removable class of $G - e$. Moreover, R contains an edge that does not lie in $M_1 \cup M_2$. This is a contradiction to Corollary 5.3 or Lemma 5.4. □

Lemma 7.7
Set $M_1 \triangle M_2$ spans a Hamiltonian cycle Q of graph $G - v' - v''$.

Proof: Let H denote the graph $G[M_1 \triangle M_2]$. As M_1 and M_2 have distinct signs in G, it follows that the number of cycles of H that have even parity in D is odd. Let Q be a cycle of H that has even parity in D. Let $M_2' := M_1 \triangle E(Q)$. Then, (D, M_1, M_2') is an e-triple of G. Every edge of $M_2' - E(Q)$ lies in M_1. By Lemma 7.6, e is the only edge of $M_1 \cap M_2'$. Thus, $M_2' - E(Q) = \{e\}$. Consequently, $V(Q) = V(G) - v' - v''$. That is, Q is a Hamiltonian cycle of $G - v' - v''$, as asserted. □

Let (U, W) be the bipartition of cycle Q. Let Z denote the set of vertices of $V(Q)$ that are adjacent to vertices in $\{v', v''\}$. As G is cubic and free of multiple edges, Z contains precisely four vertices. Each vertex of $V(Q) - Z$ is the end of precisely one chord of Q.

Lemma 7.8
No chord of Q has one end in U, the other in W.

Proof: Assume, to the contrary, that Q has a chord f that has one end in U the other end in W. (See Figure 13a.) Then, $Q + f$ has two cycles, Q_1 and Q_2. One of Q_1 and Q_2 is M_1-alternating, the other M_2-alternating. Adjust notation so that Q_i is M_i-alternating, for $i = 1, 2$. As Q has even parity in D, then precisely one of Q_1 and Q_2 has odd parity in D. Adjust notation so that Q_1 has odd parity in D. Let $M_1' := M_1 \triangle E(Q_1)$. Then, (D, M_1', M_2) is an e-triple. The edges in $E(Q_1) \cap M_2$ lie also in M_1'. Thus, $M_1' \cap M_2 - e$ is nonempty, a contradiction to Lemma 7.6. □

Lemma 7.9
Cycle Q does not have a chord having both ends in U and another chord having both ends in W.

Proof: Assume the contrary. Let $f := u_1u_2$ be a chord of Q having both ends in U, let $g := w_1w_2$ be a chord of Q having both ends in W. Consider first the case in which f and g do not cross (Figure 13(b)). In that case, f and g determine two

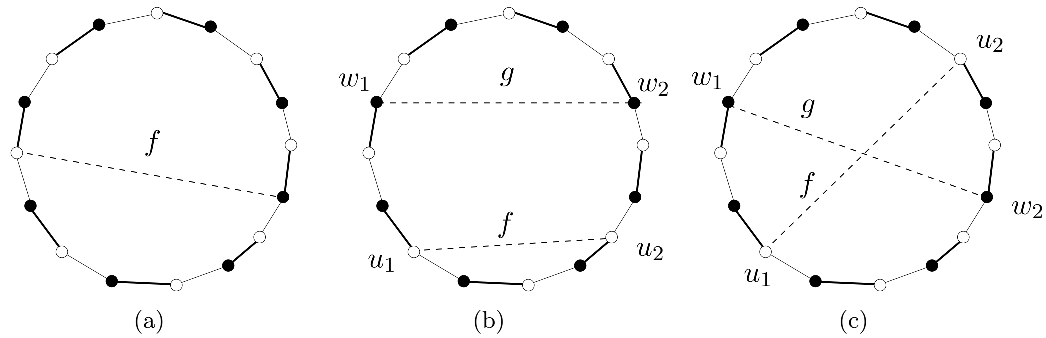


Figure 13: Proof of the Main Theorem – Chords of Q

disjoint odd cycles C_1 and C_2 in Q such that the complement of $C_1 \cup C_2$ in G has a perfect matching. By Theorem 2.4, G is nonsolid. This is a contradiction.

We may thus assume that f and g cross. Then, u_1, w_1, u_2, w_2 occur in Q in that cyclic order. (See Figure 13(c).) Let $S := \{u_1, w_1, u_2, w_2\}$. For any two cyclically consecutive distinct vertices x and y in S, let $Q[x, y]$ denote the path in Q from x to y internally disjoint with S. Let

$$\begin{aligned} Q_1 &:= Q[w_2, u_1] \cdot (u_1, f, u_2) \ \cdot Q[u_2, w_1] \cdot (w_1, g, w_2) \\ Q_2 &:= Q[u_1, w_1] \cdot (w_1, g, w_2) \cdot Q[w_2, u_2] \cdot (u_2, f, u_1). \end{aligned}$$

One of Q_1 and Q_2 is M_1-alternating, the other is M_2-alternating. Adjust notation so that Q_i is M_i-alternating, for $i = 1, 2$. For any path P in G, let $\mathrm{fw}(P)$ denote the set of forward edges of P in D. Taking into account that for any path P of odd length, if R denotes the reverse of P then $|\mathrm{fw}(P)| + |\mathrm{fw}(R)| \equiv 1 \pmod 2$, and recalling that Q has even parity in D, we deduce that $|\mathrm{fw}(Q_1)| + |\mathrm{fw}(Q_2)| \equiv 1 \pmod 2$. Consequently, precisely one of Q_1 and Q_2 has odd parity in D. Adjust notation so that Q_1 has odd parity in D. Let $M_1' := M_1 \triangle E(Q_1)$. Then, (D, M_1', M_2) is an e-triple. The edges of $E(Q_1) \cap M_2$ lie in M_1'. Thus, $M_1' \cap M_2 - e$ is nonempty, in contradiction to Lemma 7.6. □

We now derive the final contradiction, arriving to the conclusion that G is a nonsolid brick. We know that G has eight or more vertices. Therefore, Q has at least one chord, say $f := u_1u_2$. By Lemma 7.8, u_1 and u_2 are both in U or in W. Adjust notation so that u_1 and u_2 are both in U. There are two odd paths, P_1 and P_2, in Q, with ends in u_1 and u_2. By Lemma 7.5, G has no triangles. It follows that $|V(P_1)| \geq 5$. Similarly, $|V(P_2)| \geq 5$. Thus, P_1 and P_2 have each one at least two internal vertices in W. (See Figure 14(a).) By Lemmas 7.8 and 7.9, every chord of Q has both ends in U. Therefore, the vertices of W are all adjacent to vertices in $\{v', v''\}$. Consequently, $|W| = 4$. Therefore, Q has eight vertices and G has ten vertices. The internal vertices of P_1 and P_2 in U must thus be the ends of another chord of Q.

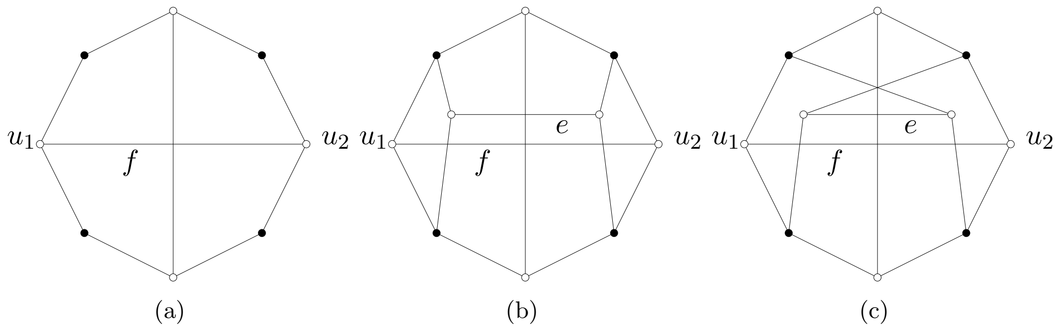


Figure 14: Graph G

The ends of e may lie in a quadrilateral in $G - e$ or not. There are precisely two graphs, up to isomorphism, correspondent to each one of these cases (Figure 14(b) and (c)). In any case, G has ten vertices and two disjoint pentagons. By Theorem 2.4, G is not solid. This is a contradiction. The proof of the Main Theorem is complete. □

Using the techniques developed in this paper, we have been able to derive an alternative proof of the theorem due to Fischer and Little [9] which characterizes near-bipartite non-Pfaffian matching covered graphs.

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List of Assertions

Problem 1.1 {`prb:Pfaffian-recognition`} 3

(The Pfaffian Recognition Problem) Given a digraph D, decide whether D is Pfaffian.

Problem 1.2 {`prb:Pfaffian-orientation`} 3

(The Pfaffian Orientation Problem) Given a graph G, decide whether G has a Pfaffian orientation.

Lemma 1.3 5

Let M and N be two perfect matchings of a directed graph D and let ℓ denote the number of (M,N)-alternating cycles that are evenly oriented. Then, $\mathrm{sgn}(M)\,\mathrm{sgn}(N) = (-1)^{\ell}$.

Corollary 1.4 5

Let D be a directed graph and let M a perfect matching of D. Then, D is Pfaffian if and only if each M-alternating cycle of D is oddly oriented.

Corollary 1.5 5

A digraph D is Pfaffian if and only if each conformal cycle in D is oddly oriented.

Proposition 1.6 {`prp:conformal-subgraph`} 5

A matching covered graph G is Pfaffian if and only if each of its conformal subgraphs is Pfaffian.

Proposition 2.1 {`prp:splicing-separating`} 5

Any splicing of two matching covered graphs is also matching covered.

Proposition 2.2 {`prp:separating`} 6

Let G be a matching covered graph. A cut C of G is separating if and only if every edge of G lies in a perfect matching that contains precisely one edge in C.

Theorem 2.3 6

Any two applications of the tight cut decomposition procedure produce the same family of bricks and braces, up to multiple edges.

Theorem 2.4 {`thm:usp`} 7

(Reed and Wakabayashi) A brick G has a nontrivial separating cut if and only if it has two disjoint odd cycles C_1 and C_2 such that $G - (V(C_1)) \cup V(C_2))$ has a perfect matching.

Proposition 2.5 {`prp:separating-Pfaffian`} 7

Let G be a matching covered graph. If G is Pfaffian then for any separating cut $C := \partial(X)$ of G, graph G has a Pfaffian orientation D such that the two C-contractions of D are also Pfaffian.

Theorem 2.6 {thm:tight-Pfaffian} 9

A matching covered graph G is Pfaffian if and only if each of its bricks and braces is Pfaffian.

Theorem 3.1 {thm:monotonic} 9

Let R be a removable class of a matching covered graph G. Then, $b(G-R) \geq b(G)$ if R is a singleton, and $b(G-R) = b(G) - 1$ if R is a doubleton.

Theorem 3.2 {thm:rem-doubl-nearbrick} 10

For every removable doubleton $\{e, f\}$ of a brick G, the graph $G - \{e, f\}$ is a bipartite matching covered graph, e joins two vertices in one part of the bipartition of $G - \{e, f\}$ and f two vertices in the other part.

Theorem 3.3 {thm:rem-solid} 10

Every removable edge of a solid matching covered graph is b-invariant.

Theorem 3.4 {thm:ear-decomp} 10

Let G be a matching covered graph and let H be a matching covered subgraph of G. Then, H is a conformal matching covered subgraph of G if and only if there exists a sequence $(G_1, G_2, \ldots, G_r)$ of subgraphs of G such that $G_1 = G$, $G_r = H$ and, for $2 \leq i \leq r$, G_i is obtained from G_{i-1} by deleting either a removable ear or a removable double ear of G_{i-1}.

Theorem 4.1 12

A matching covered graph is Pfaffian if and only if all its minors are Pfaffian.

Theorem 4.2 {thm:little} 12

The only minimal non-Pfaffian bipartite matching covered graph is the brace $K_{3,3}$.

Theorem 4.3 {thm:main-theorem} 12

(The Main Theorem) Every minimal non-Pfaffian brick must have a nontrivial separating cut.

Corollary 4.4 {cor:main} 12

The only minimal non-Pfaffian solid matching covered graph is the brace $K_{3,3}$.

Theorem 4.5 {thm:decorations} 13

A solid matching covered graph is Pfaffian if and only if it does not contain $K_{3,3}$ and none of the three graphs shown in Figure 7 as a conformal minor.

Lemma 5.1 {lem:triple} 13

Let G be a minimal non-Pfaffian matching covered graph, e a removable edge of G. Then, G has an orientation D and two perfect matchings, M_1 and M_2, such that (i) $D - e$ is a Pfaffian orientation of $G - e$, (ii) edge e lies in $M_1 \cap M_2$, and (iii) M_1 and M_2 have distinct signs in D.

Theorem 5.2 {`thm:extension`} 14

Let G be a matching covered graph, R be a b-invariant class of G, and $\overrightarrow{G-R}$ be any Pfaffian orientation of $G-R$. Then G is Pfaffian if and only if there is an extension $\overrightarrow{G}$ of $\overrightarrow{G-R}$ which is a Pfaffian orientation of G.

Corollary 5.3 {`cor:invInM1cupM2`} 14

Let G be an minimal non-Pfaffian matching covered graph, e a b-invariant edge of G, and let D an orientation of G, M_1 and M_2 perfect matchings of G such that (D, M_1, M_2) is an e-triple. Then, every b-invariant edge of $G-e$ lies in $M_1 \cup M_2$.

Lemma 5.4 {`lem:dblInM1cupM2`} 14

Let G be a minimal non-Pfaffian brick, e a b-invariant edge of G, and let D an orientation of G, M_1 and M_2 perfect matchings of G such that (D, M_1, M_2) is an e-triple. Then, every removable doubleton of $G-e$ is a subset of $M_1 \cup M_2$.

Lemma 6.1 {`lem:peripheral`} 16

Let G be a matching covered graph free of nontrivial tight cuts, let e be a b-invariant edge of G, $C := \partial(X)$ a nontrivial cut of G such that $C-e$ is tight in $G-e$. Then, C is peripheral and precisely one of $(C-e)$-contractions of $G-e$ is bipartite. Let $J := (G-e)/\overline{X} \to \overline{x}$ be bipartite. Then, either both ends of e lie in the inner part of J or one end of e lies in the inner part of J, the other end lies in $\overline{X}$. □

Lemma 6.2 {`lem:three-case, fig:three-case`} 16

(The Three Case Lemma) Let G be a brick, e a b-invariant edge of G such that $G-e$ is not a brick. Let H be the brick of $G-e$, obtained by a tight cut decomposition of $G-e$. Then, one of the following three alternatives holds (see Figure 8):

(i) either G has a peripheral cut $C_1 := \partial(X_1)$ such that $J_1 := (G-e)/\overline{X_1} \to \overline{x_1}$ is bipartite, $H = (G-e)/X_1 \to x_1$ and edge e has one end in the inner part of J_1, the other end in $V(H) - x_1$,

(ii) or G has two peripheral cuts $C_i := \partial(X_i)$, for $i = 1, 2$, such that X_1 and X_2 are disjoint, $J_i := (G-e)/\overline{X_i} \to \overline{x_i}$ is bipartite, $H = ((G-e)/X_1 \to x_1)/X_2 \to x_2$ and edge e has one end in the inner part of J_1, the other end in the inner part of J_2,

(iii) or G has a peripheral cut $C_1 := \partial(X_1)$ such that $J_1 := (G-e)/\overline{X_1} \to \overline{x_1}$ is bipartite, $H = (G-e)/X_1 \to x_1$ and edge e has both ends in the inner part of J_1.

Theorem 6.3 {`thm:thin-brick`} 18

(The thin edge theorem for bricks) Every brick distinct from K_4, $\overline{C_6}$ and the Petersen graph has a thin edge. □

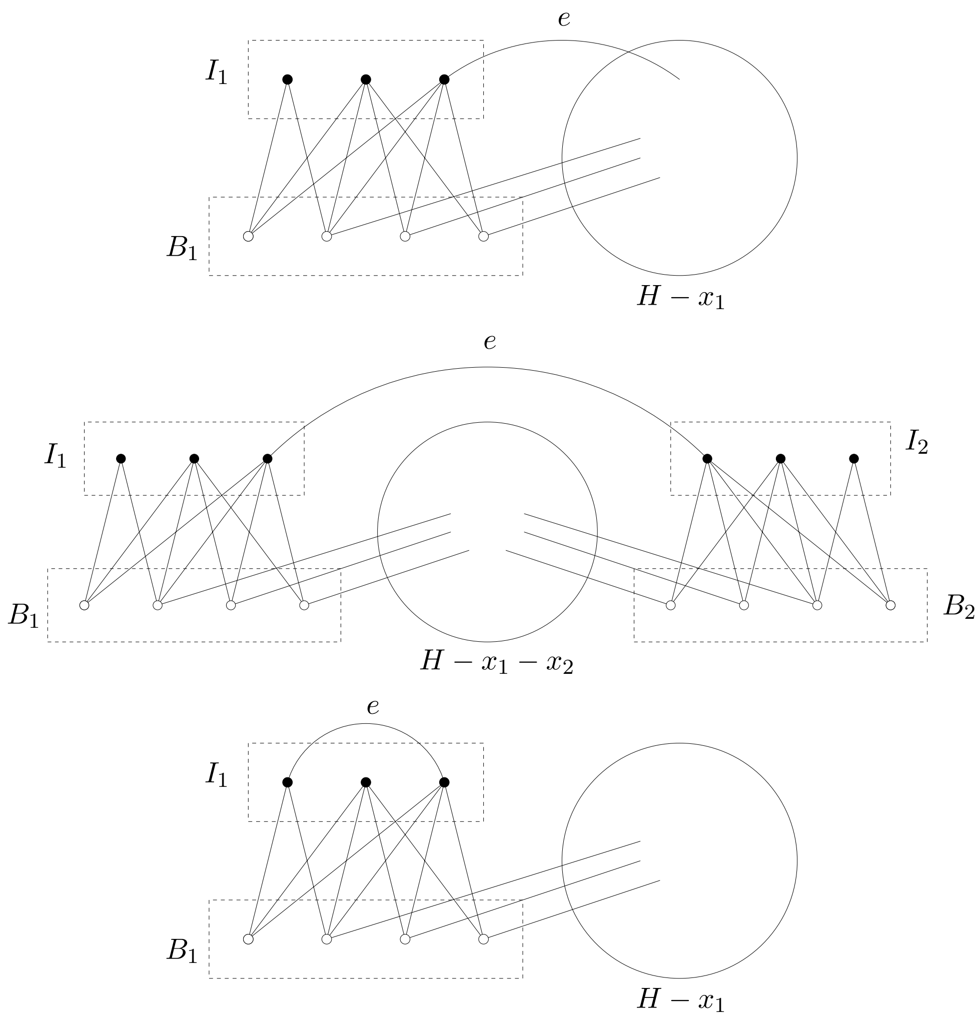

Figure 8: The three cases of Lemma 6.2

Proposition 6.4 {`prp:nonremovable-bip`} 18

(see [14]) Let G be a bipartite matching covered graph with bipartition (A, B), and let e be an edge of G. Then, e is not removable in G if and only if there is a partition (A', A'') of A and a partition (B', B'') of B with $|A'| = |B'|$ such that e is the only edge joining a vertex in A' to a vertex in B''.

Lemma 6.5 {`lem:almost-all`} 19

Let G be a matching covered graph free of nontrivial tight cuts, e a removable edge of G, $C := \partial(X)$ a peripheral cut of G such that the $(C - e)$-contraction $J := (G - e)/\overline{X} \to \overline{x}$ is bipartite. The following properties hold: (i) every edge of $C - e$ is removable in J and (ii) for any inner vertex v of J having degree three or more, at most one edge of J in $\partial(v)$ is not removable.

Corollary 6.6 {`cor:almost-all`} 20

Let G be a graph free of nontrivial tight cuts, e a removable edge of G, $C := \partial(X)$ a peripheral cut of G, let $J := (G-e)/\overline{X} \to \overline{x}$ be a bipartite $(C-e)$-contraction of $G-e$. Then, every removable class of $(G-e)/X \to x$ is also removable in $G-e$. $\square$

Theorem 6.7 20

(Tutte's Perfect Matching Theorem [20]) A graph G has a perfect matching if and only if $|\mathcal{O}(G-S)| \leq |S|$, for each subset S of $V(G)$.

Corollary 6.8 {`cor:admissible`} 20

Let G be a graph which has a perfect matching. Then the following properties hold:

(i) An edge e of G is admissible if, and only if, there is no barrier which contains both ends of e.

(ii) For each maximal barrier B of G, all components of $G-B$ are critical.

Theorem 6.9 21

(Dulmage-Mendelson Decomposition Theorem [14]) Let G be a graph with a perfect matching and bipartition (A, B). Then, there exists a partition $(A_1, A_2, \ldots, A_r)$ of A and a partition $(B_1, B_2, \ldots, B_r)$ of B, where $r \geq 1$ and such that for $i = 1, \ldots, r$, (i) the subgraph G_i of G induced by $A_i \cup B_i$ has bipartition (A_i, B_i) and is matching covered and (ii) every edge of G incident with some vertex of A_i is also incident with some vertex of B_j, where $j \leq i$.

Theorem 6.10 {`thm:canonicalBarrier, item:Hmc, item:eInOdd,`
`item:separating`} 21

Let G be a bicritical graph, and let e be a non-removable edge of G. Then, $G-e$ contains a barrier B that satisfies the following properties:

(i) Bipartite graph $H(B)$ associated with barrier B is matching covered.

(ii) Edge e has its ends in distinct odd components of $G-e-B$.

(iii) For each odd component K of $G-e-B$, cut $C(K) := \partial_G(V(K))$ is separating in G.

Lemma 6.11 {`lem:smallCap`} 23

Let G be a brick, and let e_1 and e_2 be two adjacent non-removable edges of G. Suppose that, for $i = 1, 2$, B'_i is a barrier of $G - e_i$. Then $|B'_1 \cap B'_2| \leq 1$.

Corollary 6.12 {`cor:canonicalBarrier`} 23

Let G be a solid brick, e a non-removable edge of G. Then, $G-e$ contains a barrier B such that every odd component of $G-e-B$ is trivial and edge e has its ends in distinct odd components of $G-e-B$. Moreover, the graph $H(B)$ obtained from $G-e$ by removing the vertices in even components of $G-e-B$ and the edges having both ends in B is matching covered.

Theorem 6.13 {`thm:solid:technical, fig:B-I`} 24

Let G be a solid brick, v a vertex of G, n the number of neighbours of v, d the degree of v. Enumerate the d edges of $\partial(v)$ as $e_i := vv_i$, for $i = 1, 2, \ldots, d$. Assume that neither e_1 nor e_2 is removable in G. Then, $n = 3$ and, for $i = 1, 2$, there exists an equipartition (B_i, I_i) of $V(G)$ such that

(i) e_i is the only edge of G that has both ends in I_i,

(ii) every edge that has both ends in B_i is incident with v_3, and

(iii) the bipartite subgraph H_i of G, obtained by the removal of e_i and each edge having both ends in B_i, is matching covered.

Moreover, $B_1 = (I_2 - v) \cup \{v_3\}$ and $B_2 = (I_1 - v) \cup \{v_3\}$. (See Figure 12 for an illustration.)

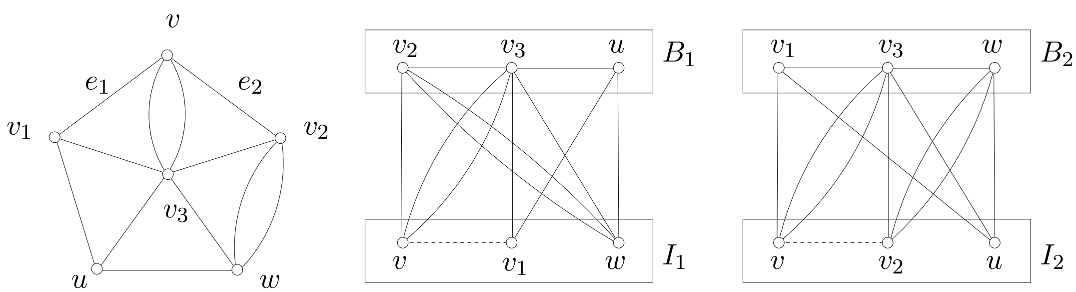

Figure 12: Graphs G, $G - e_1$ and $G - e_2$

Corollary 6.14 {`cor:atMostTwo`} 25

If G is a solid brick with six vertices or more then for every vertex v of G at most two edges incident with v are non-removable in G.

Corollary 6.15 {`cor:atMostOne`} 25

If G is a solid brick of maximum degree three or four, then, for every vertex v of G at most one edge incident with v does not lie in a removable class of G.

Lemma 7.1 {`lem:monot:lambda`} 26

If G is a solid matching covered graph and e is a removable edge of G then $G - e$ is also solid.

Lemma 7.2 26

Brick G is simple and has eight or more vertices. Moreover, it has a thin edge.

Lemma 7.3 {`lem:Hsolid`} 26

For every removable edge e of G, the brick H of $G - e$ is solid.

Lemma 7.4 26

Brick G is cubic.

Lemma 7.5 {`lem:noTriangles`} 27

Brick G is free of triangles and edge e has index two.

Lemma 7.6 {`lem:only:e`} 27

$M_1 \cap M_2 = \{e\}$.

Lemma 7.7 28

Set $M_1 \triangle M_2$ spans a Hamiltonian cycle Q of graph $G - v' - v''$.

Lemma 7.8 {`lem:oddCycle`} 28

No chord of Q has one end in U, the other in W.

Lemma 7.9 {`lem:monochrom`} 28

Cycle Q does not have a chord having both ends in U and another chord having both ends in W.