3-Flows and Minimal Combs - Supporting Material

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A Analysis of 7-Cuts

This is an appendix to the paper "3-Flows and Minimal Combs", by Cândida Nunes da Silva and Cláudio L. Lucchesi [1]. We give here details of the proof of the Main Theorem [1, Theorem 3.1], when $r = 7$ (Case 4). We adopt the notation used in the proof of the Main Theorem.

Case 4 $r = 7$.

Assume that C is a bond but not a comb. A mod 3-orientation of a 7-cut C orients five edges, called the *majority edges*, in one direction and the remaining two, called the *minority edges*, in the other direction. Therefore, the number of non-similar mod 3-orientations of C is $\binom{7}{2} = 21$. For $i = 1, 2$, we say that two edges of C are *compatible in* G_i if there is a feasible mod 3-orientation of C in G_i having these two edges as the minority edges. We define the *compatibility graph* L_i of G_i as the graph with seven vertices, each representing one edge of C, such that two edges of C are adjacent in L_i if and only if they are compatible in G_i. We emphasize that if two edges f and g of C are parallel in G_i then each edge h of $C - f - g$ is either adjacent to both f and g in L_i, or is adjacent to neither f nor g in L_i. We say that $\overline{L_i}$, the complement of L_i, is the *incompatibility graph* of G_i. We denote by ℓ_i and $\overline{\ell_i}$ the number of edges of L_i and $\overline{L_i}$, respectively.

We denote by $G := G_\star/(Z \to z)$ a generic C-contraction of $G_\star$, without specifying whether it is G_1 or G_2. Similarly, we denote by L the compatibility graph of a generic C-contraction G.

Figure 1 depicts graph γ, the only C-contraction of $G_\star$ having multiplicity three, as shown in the next result. That graph has 12 feasible non-similar mod 3-orientations.

Lemma A.1 *Let μ denote the maximum multiplicity of edges of G. Then, $\mu \leq 3$, with equality only if G is the graph γ depicted in Figure 1.*

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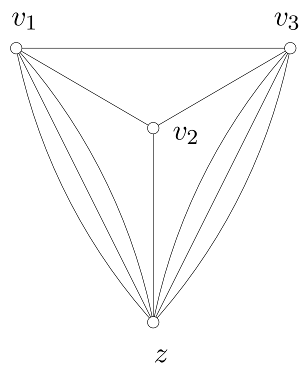

Figure 1: Graph γ is the only C-contraction of $G_\star$ with edge multiplicity three

Proof: Every vertex of $G_\star$ has degree three or five. By Lemma 2.3, $\mu \leq 3$. Assume that $\mu = 3$. Let v_1 be a vertex of G that is joined to z by three edges. Then, v_1 has degree five and is joined to $Y := \overline{Z} - v_1$ by two edges, whence $D := \partial(Y)$ is a 6-cut. Every 6-cut of $G_\star$ is acyclic. The shore $\overline{Y}$ of D in $G_\star$ includes Z. Thus, $G_\star[\overline{Y}]$ is cyclic, whence Y is the grip of D. By Corollary 2.7, the possible degree sequences of the vertices of Y are $(5, 3)$ and $(3, 3)$. Note that Y cannot have both vertices of degree three, otherwise both would be joined to z by two or more edges, a contradiction to Lemma 2.3. Thus, Y has one vertex of degree three, the other of degree five. Moreover, $G_\star[Y]$ is connected. We conclude that $G = \gamma$. □

Lemma A.2 *Graph G_2 has no 4-cuts. If C separates $S_\star$ then every 5-cut of G_1 and every 5-cut of G_2 is trivial.*

Proof: Let $G := G_\star/(Z \to z)$ denote a C-contraction of $G_\star$. Let $D := \partial(Y)$ be a cut of G, such that $4 \leq |D| \leq 5$. Assume also that either D is a non-trivial 5-cut or D is a 4-cut. Adjust notation so that $Y \subset \overline{Z}$. Cut D is a comb, its shore Y is a grip. By Corollary 2.7, if $|D| = 4$ then Y consists of two vertices, both in $S_\star$, whereas if $|D| = 5$ then Y consists of three vertices, all in $S_\star$. If $|D| = 5$ then C does not separate $S_\star$. If $|D| = 4$ then $|S_\star \cap V(G)| \geq 2$, whence $G \neq G_2$. □

Case 4.1 *Cut C does not separate $S_\star$.*

Lemma A.3 $\ell_1 \geq 3$.

Proof: By hypothesis, $|S_1| = 3$, whence $S_1 = S_\star$ and G_1 is not γ. The multiplicity μ of edges of G_1 satisfies $\mu \leq 2$. Let $\mathcal{D} := \{D_1, D_2, \ldots, D_r\}$ be a collection of non-similar mod 3-orientations of G_1. Graph G_1 has a mod 3-orientation, D_1. Thus, $r \geq 1$. Suppose $r < 3$. We will show that G_1 has a mod 3-orientation that is not similar to any of the orientations in $\mathcal{D}$. Note that three or more edges of C are majority edges in all orientations of $\mathcal{D}$. In fact, if $r = 1$ five of them are majority edges and if $r = 2$ at least three of them are majority edges on both orientations. Adjust notation so that edges $e_i := \overline{x}v_i$, $i = 1, 2, 3$, are majority edges in all mod 3-orientations of $\mathcal{D}$.

Let $T := \{e_1, e_2, e_3\}$. As G_1 and γ are distinct, then at least two edges in T are not parallel in G. Adjust notation so that e_1 and e_2 are not parallel. Let H_{12} be the graph obtained from G_1 by splitting the contraction vertex $\overline{x}$ of G_1 on e_1 and e_2. Assume that H_{12}, together with $S_\star$, does not satisfy the hypothesis of the Conjecture. By Lemma 2.4, G_1 has a 5-cut D_{12} that contains both e_1 and e_2 but does not separate $S_\star$. As D_{12} contains both e_1 and e_2, it follows that D_{12} is non-trivial. By Lemma 2.1, D_{12} is a comb and its grip Y_{12} consists of the three vertices of degree three of $S_\star$. Whence, v_1 and v_2 are vertices of $S_\star$. By Lemma 2.3, they are joined to $\overline{x}$ by one single edge. Then, e_3 is not parallel with any of e_1 and e_2. Repeating the reasoning above with e_3 playing the role of e_2, we deduce that G_1 has a 5-comb D_{13} that contains e_3 and whose grip consists of the three vertices of $S_\star$. We conclude that $S_\star = \{v_1, v_2, v_3\}$. Moreover, $D_{12} = D_{13}$. Therefore, the three edges of T lie in a 5-comb of G_1 whose grip is $S_\star$. This is a contradiction, as every mod 3-orientation of G_1 orients two of the edges of T in one direction, the third edge in the other direction (Figure 2).

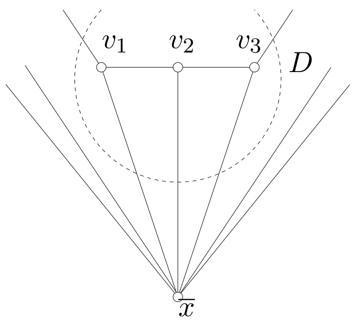

Figure 2: Illustration for the proof of Lemma A.3

Assume, without loss of generality, that H_{12}, together with $S_\star$, satisfies the hypothesis of the Conjecture. Graph H_{12} has as many edges as G_1, which in turn has fewer edges than $G_\star$. Thus, H_{12} has a mod 3-orientation, D_2. Therefore, D_2 is a mod 3-orientation of G_1 such that one of e_1 and e_2 is a minority edge. Hence, D_2 is not similar to any of the mod 3-orientations in $\mathcal{D}$. We conclude that G_1 has at least three non-similar mod 3-orientations, as asserted. □

Lemma A.4 $\overline{\ell_2} \leq 2$.

Proof: In the proof of this assertion, we use Lemmas A.5 and A.6 shown below.

Lemma A.5 *The edges of the incompatibility graph* $\overline{L_2}$ *are pairwise adjacent.*

Proof: Let P_1 and P_2 be two disjoint pairs of edges of C. We must show that at least one of P_1 and P_2 is compatible, that is, there exists a mod 3-orientation of G_2 such that one of P_1 and P_2 is the pair of minority edges. For this, assume, without loss of generality,

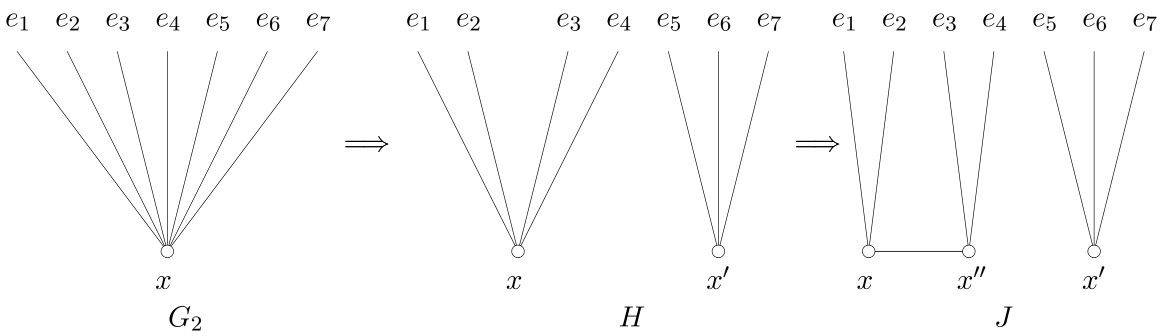

Figure 3: Graphs H and J in the proof of Lemma A.5

that $P_1 = \{e_1, e_2\}$ and $P_2 = \{e_3, e_4\}$. Let $P := P_1 \cup P_2$, $P' := \{e_5, e_6, e_7\}$. Let H be the graph obtained from G_2 by splitting x on $\{P, P'\}$ (Figure 3). Let x' denote the new vertex of H. Let J be the graph obtained from H by expanding x on e_3 and e_4. Let x'' denote the new vertex of J (Figure 3). Let $S_J := \{x, x', x''\}$. We assert that J and S_J satisfy the hypothesis of the Conjecture.

Cut C is a bond, therefore $G_\star[\overline{X}]$ is connected. Thus, J is connected and edge xx'' is not a bridge. Assume, to the contrary, that J has an 1-cut C_1. Then C_1 separates $\{x, x'\}$ but is not edge xx'', whence $C_1 \cup \{e_5, e_6, e_7\}$ is a 4-cut of G_2, a contradiction to Lemma A.2. Thus, J is 2-edge-connected. Graph G_2 is free of vertices of degree three, therefore every 3-cut of J separates S_J. We deduce that J and S_J satisfy the hypothesis of the Conjecture.

Finally, J has one more edge than G_2, which in turn has no more than $|E(G_\star)| - 3$ edges. Thus, J has fewer edges than $G_\star$, whence it has a mod 3-orientation. Consequently, G_2 has a mod 3-orientation in which one of P_1 and P_2 is the pair of minority edges. Thus, one of P_1 and P_2 is compatible. This conclusion holds for each pair P_1, P_2 of disjoint pairs of edges of C. As asserted, any two edges of $\overline{L_2}$ are adjacent. □

Lemma A.6 *For every pair P of non-parallel edges of C, every edge of $C - P$ is adjacent in L_2 to at least one edge in P.*

Proof: Adjust notation so that $P = \{e_1, e_2\}$. Let $S := \{x\} = S_\star/(X \to x)$. Let H be the graph resulting from G_2 by the splitting of x on e_1 and e_2 (Figure 4). Let w be the new vertex of H. Let J be the graph obtained from H by expanding x on e_4 and e_5 and then expanding again on e_6 and e_7. Let x' and x'' denote the two new vertices of J, where x' is incident with e_4 and e_5, and x'' is incident with e_6 and e_7 (Figure 4). Let $S_J := \{x, x', x''\}$. By Corollary 2.5, H and S satisfy the hypothesis of the Conjecture. Moreover, as C is a bond, vertex x is not a cut vertex of H. Thus, J and S_J also satisfy the hypothesis of the Conjecture. Graph J has two more edges than G_2, which in turn has no more than $|E(G_\star)| - 3$ edges. Thus, J has a mod 3-orientation. We conclude that one of e_1 and e_2 is adjacent to e_3 in L_2. This conclusion holds for each edge e_3 in $C - P$. □

We may now resume the proof of Lemma A.4. Graphs G_2 and γ are distinct. Suppose to the contrary that $\overline{L_2}$ has at least three edges. By Lemma A.5, all edges of $\overline{L_2}$ are pairwise

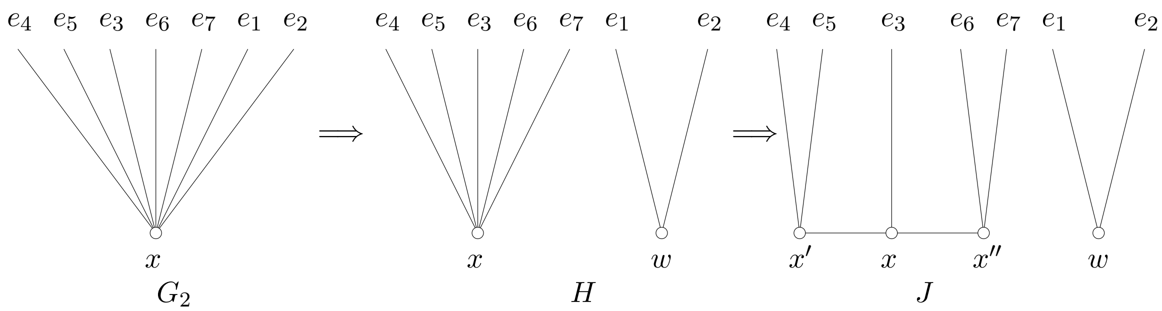

Figure 4: Graphs H and J in the proof of Lemma A.6

adjacent. Thus, either $\overline{L_2}$ has a triangle or a three pointed star. Consider first the case in which $\overline{L_2}$ has a triangle. Adjust notation so that e_1e_2, e_1e_3 and e_2e_3 are the edges of the triangle. Then, by Lemma A.6, the three edges e_1, e_2 and e_3 of cut C are parallel in G_2, a contradiction, as G_2 is not γ. We may thus assume that $\overline{L_2}$ has a three pointed star. Adjust notation so that e_1e_2, e_1e_3 and e_1e_4 are the edges of the star. Then, by Lemma A.6, the three edges e_2, e_3 and e_4 of cut C are parallel in G_2, again a contradiction. In fact, $\overline{L_2}$ has at most two edges. □

We are now in condition to prove the Theorem for this case. By Lemma A.3, $\ell_1 \geq 3$. By Lemma A.4, $\ell_2 \geq 19$. Thus, $\ell_1 + \ell_2 > 21$, whence C has a mod 3-orientation that is feasible in both G_1 and G_2. Consequently, $G_\star$ has a mod 3-orientation, a contradiction.

Case 4.2 *Cut C separates $S_\star$.*

Lemma A.7 (Double Splitting) *Let $G := G_\star/(Z \to z)$ be a generic C-contraction of $G_\star$. Consider the pair $\mathcal{P} := \{P_1, P_2\}$, where P_1 and P_2 are disjoint pairs of edges of C. Let m denote the number of pairs P_i in $\mathcal{P}$ that consist of parallel edges. If $m + |S_\star - Z| \leq 2$ then the compatibility graph L has an edge joining an edge of C in P_1 to an edge of C in P_2.*

Proof: By hypothesis, $|S_\star - Z| \geq 1$. Thus, $m \leq 1$. We may thus adjust notation so that the edges in P_2 are not parallel. Adjust notation so that $P_1 = \{e_1, e_2\}$ and $P_2 = \{e_3, e_4\}$. Define S to be

$$S := \begin{cases} S_\star/(Z \to z), & \text{if } v_1 \neq v_2, \\ \{v_1\} \cup (S_\star/(Z \to z)), & \text{otherwise.} \end{cases}$$

Note that the contraction vertex z lies in $S_\star/(Z \to z)$. Thus,

$$|S| = m + |S_\star/(Z \to z)| = m + |S_\star - Z| + 1 \leq 3.$$

Let H denote the graph obtained from G by splitting z on e_1 and e_2. Let z' denote the new vertex of H (Figure 5). Let J be the graph obtained from H by splitting z on e_3 and e_4. Let z'' denote the new vertex of J.

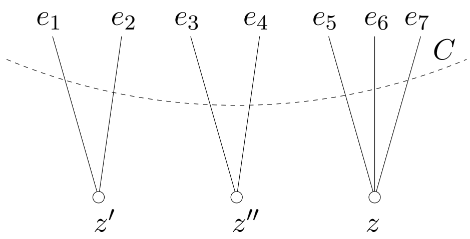

Figure 5: A double splitting

Claim *Graph J, together with S, satisfies the hypothesis of the Conjecture.*

Proof: Assume the contrary. By hypothesis, C is a bond, thus J is connected. We may thus assume that J has a cut D_J that is either a 3-cut that does not separate S or a 1-cut. By Corollary 2.5, graph H, together with S, satisfies the hypothesis of the Conjecture. Thus, D_J is not a cut of H. We conclude that H has a cut D_H that includes P_2 and such that either D_H is a 3-cut or D_H is a 5-cut that does not separate S.

We assert that G contains a cut D that includes P_2 and either it is a 7-cut that does not separate S or it is 5-cut of G. For this, consider first the case in which D_H is a cut of G. As D_H includes P_2, in turn a pair of non-parallel edges, D_H cannot be a 3-cut. In that case, D_H is a 5-cut of G that includes P_2. Alternatively, consider next the case in which D_H is not a cut of G. In that case, $D_H \cup P_1$ is a cut of G. Moreover, if D_H does not separate S then neither does $D_H \cup P_1$. We conclude that G has a cut D such that either D is a 5-cut that includes P_2 or D is a 7-cut that does not separate S. If D is a 7-cut that does not separate S then it does not separate $S_\star$ in $G_\star$, a case already considered. If D is a 5-cut that includes P_2 then it is non-trivial, in contradiction to Lemma A.2. □

Note that J has as many edges as G, which in turn has fewer edges than $G_\star$. Consequently, J has a mod 3-orientation. In every mod 3-orientation of J, one of the edges of P_1, together with an edge of P_2, constitutes the pair of minority edges of C. □

Corollary A.8 *Consider a quadruple Q of edges of C, let m denote the number of pairs of edges in Q that are parallel in G. If $m + |S_\star - Z| \leq 2$ then the compatibility graph L has two adjacent edges joining edges of C that lie in Q.*

Proof: Assume that $m + |S_\star - Z| \leq 2$. Adjust notation so that $Q = \{e_1, e_2, e_3, e_4\}$. Let $P_1 := \{e_1, e_3\}$, let $P_2 := \{e_2, e_4\}$. By the Lemma, L has an edge joining an edge of C in P_1 to an edge of C in P_2. Adjust notation so that e_1 and e_2 are adjacent in L. Let $P'_1 := \{e_1, e_2\}$, $P'_2 := \{e_3, e_4\}$. Again, by the Lemma, L has an edge joining e_i, $i \in \{1, 2\}$, to an edge of C in P'_2. □

Case 4.2.1 *Graph G_2 is not simple.*

Graph G_1 has two vertices of degree three. Thus, G_1 and γ are distinct. It follows that the maximum multiplicity of G_1 is two.

Consider first the case in which $G_2 = \gamma$. It is easy to see that C includes a quadruple Q which induces in L_2 the complete graph on four vertices. The maximum multiplicity of edges of C in G_1 is two. Therefore, there exists a partition $\{P_1, P_2\}$ of edges of Q in two pairs such that the edges of each pair are not parallel in G_1. By Lemma A.7, G_1 has a mod 3-orientation such that some edge f_1 of P_1 and some edge f_2 of P_2 constitute the pair of minority edges. But f_1 and f_2 are adjacent in L_2, therefore G_2 also has a mod 3-orientation having f_1 and f_2 as minority edges. In that case, $G_\star$ has a mod 3-orientation, a contradiction.

We may thus assume that G_2 and γ are distinct. In that case, the maximum multiplicity of edges of G_2 is two. By hypothesis, G_2 has parallel edges. Adjust notation so that e_1 and e_2 are parallel in G_2. Let Y be the set of edges of $C - e_1 - e_2$ that are adjacent to e_1 in L_2. Note that as e_1 and e_2 are parallel, then Y is also the set of edges of $C - e_1 - e_2$ that are adjacent to e_2 in L_2. We assert that $|Y| \geq 3$. For this, let $P_1 := \{e_1, e_2\}$, let T be a triple of edges in $C - e_1 - e_2$. The maximum multiplicity is two, therefore T includes a pair P_2 of edges that are not parallel in G_2. By Lemma A.7, one edge of C in P_1 is adjacent in L_2 to some edge of C in P_2. Edges e_1 and e_2 are parallel in G_2. Thus, both are adjacent in L_2 to the edge in P_2. This conclusion holds for each triple T that is a subset of $C - e_1 - e_2$. Consequently, $|Y| \geq 3$. In sum, each of e_1 and e_2 is adjacent in L_2 to each edge of a set Y of three or more edges of $C - e_1 - e_2$.

Thus, Y includes a pair P_3 of edges that are not parallel in G_1. Edges e_1 and e_2 are parallel in G_2 and $G_\star$ is simple, therefore e_1 and e_2 are not parallel in G_1. By Lemma A.7, with P_3 playing the role of P_2, we conclude that an edge in P_1 is adjacent to an edge in P_3 in L_1. We have seen that every element of P_1 is adjacent to every element of P_3 in L_2. Thus, L_1 and L_2 have an edge in common, whence $G_\star$ has a mod 3-orientation. This is a contradiction.

Case 4.2.2 *Graph G_2 is simple.*

For this case, we need to introduce a new graph, which we call *the crown*. This graph is depicted in Figure 6.

Lemma A.9 *If L_1 has a quadrilateral or the crown as a subgraph then $G_\star$ has a mod 3-orientation.*

Proof: Consider first the case in which L_1 has a quadrilateral Q as a subgraph. Adjust notation so that $Q = (e_1, e_2, e_3, e_4)$. By hypothesis, G_2 is simple. By Corollary A.8, L_2 has two adjacent edges whose ends lie in Q. One of them is an edge of L_1. Thus, L_1 and L_2 have a common edge, whence $G_\star$ has a mod 3-orientation.

Consider next the case in which L_1 has the crown as a subgraph. Let H be the graph obtained from G_2 by expanding x on e_5 and e_6, let x' denote the new vertex of H. Let J

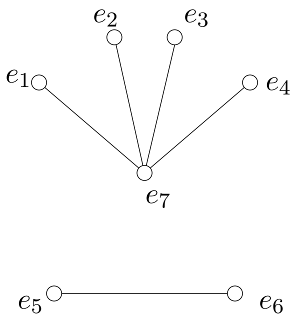

Figure 6: The crown

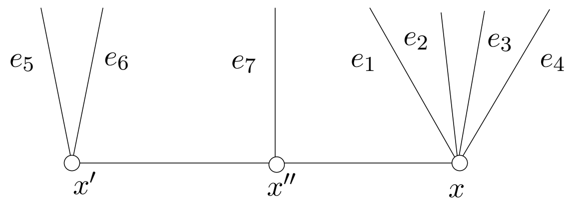

Figure 7: The graph J in the proof of Lemma A.9

be the graph obtained from H by expanding x on xx' and e_7. Let x'' denote the new vertex of J (Figure 7).

Let $S := S_2 \cup \{x', x''\}$. We assert that J and S satisfy the hypothesis of the Conjecture. Cut C is a bond, thus $G_2 - x$ is connected, whence J is connected and edges xx'' and $x'x''$ are not bridges. Moreover, $G_\star$ is 2-edge-connected. Thus, J is 2-edge-connected. Assume, to the contrary, that J has a 3-cut C_3 that does not separate S. The only 3-cut of G_2 is trivial, it has the vertex of S_2 as the only vertex of its singleton shore. That cut separates S. Thus, C_3 is not a cut of G_2. Consequently, it contains at least one of the edges xx'' and $x'x''$. If it contains edge $x'x''$ then it separates S. We may thus assume that C_3 contains only edge xx'' not in $E(G_2)$. Let $D := C_3 \triangle \partial_J(x)$. Vertex x does not lie in S, therefore D is a cut of G_2 that does not separate S. Moreover, it is a k-cut, for $k \in \{2, 4, 6\}$. Certainly $k \neq 2$. Also, $k \neq 4$, by Lemma A.2. Thus, $k = 6$. Let $Y \subset \overline{X}$ be a shore of C_6. We have seen that every 6-cut of $G_\star$ is acyclic. By Corollary 2.7, D separates $S_\star$. Thus, Y contains the only vertex of S_2, whence D separates S. This is a contradiction.

Graph J has two more edges than G_2, which, in turn has no more than $|E(G_\star)| - 3$ edges. Thus, J has a mod 3-orientation. Consequently, G_2 has a mod 3-orientation in which either e_5 and e_6 are the minority edges, or e_7 is a minority edge, together with one of e_1, e_2, e_3 and e_4. In both alternatives, L_2 has an edge in common with L_1, whence $G_\star$ has a mod 3-orientation. □

To complete the proof, recall that graph G_1 is not γ, because it contains two vertices of degree three. Thus, the maximum multiplicity of G_1 is two.

Lemma A.10 *If G_1 is simple then $G_\star$ has a mod 3-orientation.*

Proof: We shall prove the assertion by proving that L_1 has either a quadrilateral or the crown as a subgraph. Let Δ denote the maximum degree of vertices of L_1. Adjust notation so that e_7 has degree Δ in L_1. Let e_i, $i = 1, 2, \ldots, \Delta$ denote the neighbours of e_7.

Consider first the case in which $\Delta \leq 2$. By Corollary A.8, every quadruple of vertices of L_1 spans at least a pair of adjacent edges. Thus, $\Delta \geq 2$, whence $\Delta = 2$. Consider the quadruple $Q_1 := \{e_3, e_4, e_5, e_6\}$. If Q_1 spans a complete graph then L_1 has a quadrilateral as a subgraph. We may thus assume that Q_1 does not span a complete graph. Adjust notation so that e_3 and e_4 are not adjacent. Consider the quadruple $Q_2 := \{e_1, e_2, e_3, e_4\}$. Vertex e_1 cannot be adjacent to two vertices in Q_2, otherwise $\Delta \geq 3$, a contradiction. Likewise, e_2 is not adjacent to two vertices of Q_2. By Corollary A.8, one of e_3 and e_4 is adjacent to two vertices of Q_2. Adjust notation so that e_3 is adjacent to two vertices of Q_2. As e_3 and e_4 are not adjacent, then e_3, just like e_7, is adjacent to both e_1 and e_2. Thus, L_1 has a quadrilateral. Consider next the case in which $\Delta = 3$. Let $Q_3 := \{e_7, e_1, e_4, e_5\}$. Vertex e_7 is not adjacent to e_4, nor to e_5. By Lemma A.7, e_1 is adjacent to at least one of e_4 and e_5. Repeating this reasoning twice, once with e_2 playing the role of e_1, then e_3 playing the role of e_1, we deduce that each vertex in $\{e_1, e_2, e_3\}$ is adjacent to at least one of e_4 and e_5. It follows that one of e_4 and e_5 is adjacent to at least two vertices in $N(e_7)$. Thus, L_1 has a quadrilateral.

Finally, assume that $\Delta \geq 4$. If e_5 and e_6 are adjacent then L_1 has the crown as a subgraph. We may thus assume that e_5 and e_6 are not adjacent. Consider the quadruple $Q_4 := \{e_1, e_2, e_5, e_6\}$. If e_5 is adjacent to e_1 and to e_2 then L_1 has a quadrilateral. We may thus assume that e_5 is adjacent to at most one vertex of Q_4. Likewise, we may assume that e_6 is adjacent to at most one vertex of Q_4. By Corollary A.8, one of the vertices in $\{e_1, e_2\}$ is adjacent to two vertices of Q_4. Adjust notation so that e_1 is adjacent to two vertices of Q_4. We deduce that e_1 is adjacent to at least one vertex in $\{e_5, e_6\}$. By repeating a similar reasoning with other quadruples we deduce that each vertex in $\{e_1, e_2, e_3\}$ is adjacent to some vertex in $\{e_5, e_6\}$. Then, one of the vertices in $\{e_5, e_6\}$ is adjacent to two or more vertices in $\{e_1, e_2, e_3\}$, whence L_1 has a quadrilateral. □

Lemma A.11 *If G_1 is not simple then $G_\star$ has a mod 3-orientation.*

Proof: Let λ denote the number of pairs of edges of C that are parallel in G_1. Again, we show that L_1 has either a quadrilateral or the crown as a subgraph.

Consider first the case in which $\lambda = 1$. Adjust notation so that e_1 and e_2 are parallel in G_1. If e_1 is adjacent to two vertices in $C - e_1 - e_2$ then the same two vertices are adjacent to e_2, whence L_1 has a quadrilateral. We may thus assume that e_1 is adjacent to at most one vertex in $C - e_1 - e_2$. Adjust notation so that e_1 is not adjacent to e_i, for $i = 3, 4, 5, 6$. If $\{e_3, e_4, e_5, e_6\}$ spans a complete graph then L_1 has a quadrilateral. We may thus assume that it does not span a complete graph. Adjust notation so that e_3 and e_4 are not adjacent. Consider the quadruple $\{e_1, e_3, e_4, e_5\}$. By Corollary A.8, e_5 is adjacent to both e_3 and e_4. Likewise, e_6 is adjacent to both e_3 and e_4. Thus, L_1 has a quadrilateral.

Consider next the case in which $\lambda \geq 2$. Adjust notation so that e_1 and e_2 are parallel in G_1 and e_3 and e_4 are also parallel in G_1. If e_1 is adjacent to e_3 then $\{e_1, e_2, e_3, e_4\}$ spans a quadrilateral (possibly with diagonals). We may thus assume that no vertex in $\{e_1, e_2\}$ is adjacent to any vertex in $\{e_3, e_4\}$.

Let $T := \{e_5, e_6, e_7\}$. Consider next the case in which a vertex in T, say, e_5, is adjacent to both e_1 and e_3 in L_1. Then, e_5 is adjacent to each of the four vertices in $\{e_1, e_2, e_3, e_4\}$. If vertex e_3 is adjacent to both e_6 and e_7, then (e_3, e_6, e_4, e_7) is a quadrilateral of L_1. We may thus assume that e_3 and e_6 are not adjacent. By Lemma A.7, with $\mathcal{P}_1 := \{e_1, e_6\}$, and $P_2 := \{e_3, e_7\}$, we have that e_6 and e_7 are adjacent, whence L_1 has the crown as a subgraph.

We may thus assume that no vertex in T is adjacent to both e_1 and e_3. Consider next the case in which $\lambda = 3$. Adjust notation so that e_5 and e_6 are parallel. We may assume that no edge of L_1 has both ends in $\{e_1, e_3, e_5\}$, otherwise L_1 has a quadrilateral. Consider the quadruple $\{e_1, e_3, e_5, e_7\}$. By Corollary A.8, e_7 is adjacent to at least two vertices in $\{e_1, e_3, e_5\}$. We may adjust notation so that e_7 is adjacent to both e_1 and e_3. Thus, T has a vertex adjacent to both e_1 and e_3. This is a case that we have already considered. The assertion holds if $\lambda = 3$.

We may thus assume that $\lambda = 2$. For any two distinct vertices v and w in T, by considering the quadruple $\{e_1, e_3, v, w\}$, we observe that v and w are adjacent and one of v and w is adjacent to one of e_1 and e_3. Let us derive some consequences of this observation. The set T spans a triangle in L_1. Adjust notation so that e_5 is adjacent to e_1. One of e_6 and e_7 is adjacent to a vertex in $\{e_1, e_3\}$. Adjust notation so that e_6 is adjacent to a vertex in $\{e_1, e_3\}$. If e_1 is adjacent to any vertex in $T - e_5$ then L_1 has a quadrilateral. We may assume that e_5 is the only neighbour of e_1 in T and e_6 is the only neighbour of e_3 in T (Figure 8).

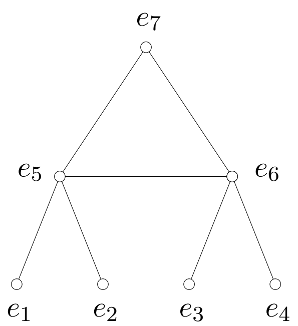

Figure 8: Illustration for the proof of Lemma A.11

Consider the pairs $\mathcal{P}_1 := \{e_1, e_3\}$ and $\mathcal{P}_2 := \{e_2, e_4\}$. By Lemma A.7, either e_1 is adjacent to e_2 or e_3 is adjacent to e_4. In both alternatives, L_1 has the crown as a subgraph. □

The full analysis of the case $r = 7$ completes the proof of the Main Theorem.

References

[1] C. N. da Silva and C. L. Lucchesi. 3-Flows and minimal combs. Submitted, 2013.

List of Assertions